On pointed Hopf algebras over dihedral groups

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Let k be an algebraically closed field of characteristic 0 and let D_m be the dihedral group of order $2m$ with $m = 4t$; $t \geq 3$. This talk will be based on a joint work with Fernando Fantino [2], where we classify all finite-dimensional Nichols algebras over D_m and all finite-dimensional pointed Hopf algebras whose group of group-likes is D_m, by means of the lifting method. As a byproduct we obtain new examples of finite-dimensional pointed Hopf algebras. In particular, we give an infinite family of non-abelian groups with non-trivial examples of pointed Hopf algebras over them and where the classification is completed. The difference with the case of the symmetric groups S_3 y S_4, see [1] and [3], respectively, is that each dihedral group provide an infinite family of new examples.

Bibliography

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[2] F. Fantino and G. A. García, *On pointed Hopf algebras over dihedral groups*. To appear in Pacific J. of Math.

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Let $m = 4t = 2n \geq 12$ and recall that

$$\mathbb{D}_m := \langle g, h \quad | \quad g^2 = 1 = h^m, gh = h^{-1}g \rangle.$$

Theorem [FG]

Let H be a finite-dimensional pointed Hopf algebra with $G(H) = \mathbb{D}_m$. Then H is isomorphic to

(a) $\mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m$ with $I = \{(i,k)\} \in \mathcal{I}$, $k \neq n$.

(b) $\mathfrak{B}(M_L)\#\Bbbk\mathbb{D}_m$ with $L \in \mathcal{L}$.

(c) $A_I(\lambda, \gamma)$ with $I \in \mathcal{I}$, $|I| > 1$ or $I = \{(i,n)\}$ and $\gamma \equiv 0$.

(d) $B_{I,L}(\lambda, \gamma, \theta, \mu)$ with $(I, L) \in \mathcal{K}$, $|I| > 0$ and $|L| > 0$.

Conversely, any pointed Hopf algebra of the list above is a lifting of a finite-dimensional braided Hopf algebra in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$.

where

- $\omega \in \mathbb{G}_m$ is an m-th primitive root of unity.
- $\mathcal{I} = \{I = \coprod_{s=1}^{r}\{(i_s, k_s)\} : \omega^{i_s k_s} = -1$ and $\omega^{i_s k_t + i_t k_s} = 1,$
  $1 \leq i_s < n, 1 \leq k_s < m\}.$
- $\mathcal{L} = \{L = \coprod_{s=1}^{r}\{\ell_s\} : 1 \leq \ell_1, \ldots, \ell_r < n, \text{ odd}\}$
- $\mathcal{K} = \{(I, L) : \ I \in \mathcal{I}, L \in \mathcal{L}$ and $\omega^{i\ell} = -1, k$ odd
  $\forall\ (i,k) \in I, \ell \in L\}.$
- $\lambda = (\lambda_{p,q,i,k})_{(p,q),(i,k)\in I}$, $\gamma = (\gamma_{p,q,i,k})_{(p,q),(i,k)\in I}$, $\theta = (\theta_{p,q,\ell})_{(p,q)\in I,\ell\in L}$, and $\mu = (\mu_{p,q,\ell})_{(p,q)\in I,\ell\in L}$ family of parameters in $\Bbbk$ that satisfy:

$$\lambda_{p,m-k,i,k} = \lambda_{i,k,p,m-k} \quad \text{and} \quad \gamma_{p,k,i,k} = \gamma_{i,k,p,k}.$$

$\mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m$

If $I = \{(i,k)\}, k \neq n$, then $\mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m$ is generated by g, h, x, y which satisfy

$$
\begin{aligned}
g^2 = 1 = h^m, &\qquad ghg = h^{m-1}, \\
gx = yg, \qquad hx = \omega^k xh, &\qquad hy = \omega^{-k} yh, \\
x^2 = 0, \qquad y^2 = 0, &\qquad xy + yx = 0
\end{aligned}
$$

It is a Hopf algebra with

$$
\begin{aligned}
\Delta(g) = g \otimes g, &\qquad \Delta(h) = h \otimes h, \\
\Delta(x) = x \otimes 1 + h^i \otimes x, &\qquad \Delta(y) = y \otimes 1 + h^{-i} \otimes y.
\end{aligned}
$$

$\mathfrak{B}(M_L)\#\Bbbk\mathbb{D}_m$

Let $L \in \mathcal{L}$, $\mathfrak{B}(M_L)\#\Bbbk\mathbb{D}_m$ is generated by $z_\ell, w_\ell, \ell \in L$ which satisfy:

$$g^2 = 1 = h^m, \qquad ghg = h^{m-1},$$
$$gz_\ell = w_\ell g, \qquad hz_\ell = \omega^\ell z_\ell h, \qquad hw_\ell = \omega^{-\ell} w_\ell h,$$
$$z_\ell^2 = 0, \qquad w_\ell^2 = 0, \qquad z_\ell w_{\ell'} + w_{\ell'} z_\ell = 0, \qquad z_\ell z_{\ell'} + z_{\ell'} z_\ell = 0.$$

It is a Hopf algebra with

$$\Delta(g) = g \otimes g, \qquad \Delta(h) = h \otimes h,$$
$$\Delta(z_\ell) = z_\ell \otimes 1 + h^n \otimes z_\ell, \qquad \Delta(w_\ell) = w_\ell \otimes 1 + h^n \otimes w_\ell.$$

$A_I(\lambda, \gamma)$

For any $I \in \mathcal{I}$, $A_I(\lambda, \gamma)$ is the algebra generated by $g, h, x_{p,q}, y_{p,q}$ with $(p, q) \in I$ satisfying:

$$g^2 = 1 = h^m, \qquad ghg = h^{m-1},$$
$$gx_{p,q} = y_{p,q}g, \qquad hx_{p,q} = \omega^q x_{p,q}h, \qquad hy_{p,q} = \omega^{-q} y_{p,q}h,$$
$$x_{p,q}x_{i,k} + x_{i,k}x_{p,q} = \delta_{q,m-k}\lambda_{p,q,i,k}(1 - h^{p+i}),$$
$$x_{p,q}y_{i,k} + y_{i,k}x_{p,q} = \delta_{q,k}\gamma_{p,q,i,k}(1 - h^{p-i}).$$

It is a Hopf algebra with

$$\Delta(g) = g \otimes g, \qquad \Delta(h) = h \otimes h,$$
$$\Delta(x_{p,q}) = x_{p,q} \otimes 1 + h^p \otimes x_{p,q}, \qquad \Delta(y_{p,q}) = y_{p,q} \otimes 1 + h^{-p} \otimes y_{p,q}.$$

$B_{I,L}(\lambda, \gamma, \theta, \mu)$

Let $(I, L) \in \mathcal{K}$, $B_{I,L}(\lambda, \gamma, \theta, \mu)$ is the algebra generated by $g, h, x_{p,q}, y_{p,q}, z_\ell, w_\ell$, $(p, q) \in I, \ell \in L$, satisfying: g, h as before &

$$gx_{p,q} = y_{p,q}g, \qquad hx_{p,q} = \omega^q x_{p,q}h,$$
$$gz_\ell = w_\ell g, \qquad hz_\ell = \omega^\ell z_\ell h,$$
$$x_{p,q}^2 = 0 = y_{p,q}^2 \qquad z_\ell w_{\ell'} + w_{\ell'} z_\ell = 0 \qquad z_\ell z_{\ell'} + z_{\ell'} z_\ell = 0$$
$$x_{p,q}x_{i,k} + x_{i,k}x_{p,q} = \delta_{q,m-k}\lambda_{p,q,i,k}(1 - h^{p+i}),$$
$$x_{p,q}y_{i,k} + y_{i,k}x_{p,q} = \delta_{q,k}\gamma_{p,q,i,k}(1 - h^{p-i}),$$
$$x_{p,q}z_\ell + z_\ell x_{p,q} = \delta_{q,m-\ell}\theta_{p,q,\ell}(1 - h^{n+p}),$$
$$x_{p,q}w_\ell + w_\ell x_{p,q} = \delta_{q,\ell}\mu_{p,q,\ell}(1 - h^{n+p}).$$

It is a Hopf algebra with g, h group-likes and

$$\Delta(x_{p,q}) = x_{p,q} \otimes 1 + h^p \otimes x_{p,q}, \quad \Delta(y_{p,q}) = y_{p,q} \otimes 1 + h^{-p} \otimes y_{p,q},$$
$$\Delta(z_\ell) = z_\ell \otimes 1 + h^n \otimes z_\ell, \qquad \Delta(w_\ell) = w_\ell \otimes 1 + h^n \otimes w_\ell.$$

Let H be a pointed Hopf algebra, $H_0 = \Bbbk G(H)$.

$\{H_i\}_{i\geq 0}$ *coradical filtration* of H.

Fact: If H_0 is a Hopf subalgebra, then
$\operatorname{gr} H = \oplus_{n\geq 0} \operatorname{gr} H(n)$ is a graded Hopf algebra,
$\operatorname{gr} H(n) = H_n/H_{n-1}$, $H_{-1} = 0$.

If $\pi : \operatorname{gr} H \to H_0$ denotes the homogeneous projection, then

$$R = (\operatorname{gr} H)^{\operatorname{co}\pi} = \{h \in H : \ (\operatorname{id} \otimes \pi)\Delta(h) = h \otimes 1\}$$

is the *diagram* of H; and $\operatorname{gr} H \simeq R\#\Bbbk G(H)$.

- R is a (braided) Hopf algebra in the category ${}^{H_0}_{H_0}\mathcal{YD}$ of Yetter-Drinfel'd modules over H_0.
- R is a graded subalgebra of $\operatorname{gr} H$.
- The linear subspace $R(1)$, together with the braiding of ${}^{H_0}_{H_0}\mathcal{YD}$, is called the infinitesimal braiding of H and coincides with

$$P(R) = \{r \in R : \ \Delta_R(r) = r \otimes 1 + 1 \otimes r\}.$$

- The subalgebra of R generated by $P(R) = V$ is (isomorphic to) the Nichols algebra $\mathfrak{B}(V)$.

Let G be a finite group and $H_0 = \Bbbk G$. Main steps for classifying finite-dimensional pointed Hopf algebras over G are

(a) determine all Yetter-Drinfel'd modules V such that $\mathfrak{B}(V)$ is finite-dimensional,

(b) For such V, determine all Hopf algebras H such that $\operatorname{gr} H \simeq \mathfrak{B}(V)\#H_0$, H is called a *lifting* of $\mathfrak{B}(V)$ over G.

(c) Prove that any finite-dimensional pointed Hopf algebra over G is generated by group-likes and skew-primitives.

It was introduced by Andruskiewitsch and Schneider

Complete classification of finite-dimensional pointed Hopf algebras over G (with non-trivial examples) where

- G finite and abelian with $(|G|, 210) = 1$ [AS].
- $G = \mathbb{S}_3$, [AS & Heckenberger].
- $G = \mathbb{S}_4$, [AHS] and [G. & A. García Iglesias].

Let G be a finite group. Recall that a Yetter-Drinfel'd module over $\Bbbk G$ is a G-module and a $\Bbbk G$-comodule M such that

$$\delta(g.m) = ghg^{-1} \otimes g.m, \qquad \forall\ m \in M_h, g, h \in G,$$

where $M_h = \{m \in M : \delta(m) = h \otimes m\}$, $M = \oplus_{h \in G} M_h$.

Proposition

- Finite-dimensional Yetter-Drinfel'd modules over G are completely reducible.
- Irreducible modules are parametrized by pairs $(\mathcal{O}, \rho)$, where $\mathcal{O}$ is a conjugacy class of G and (ρ, V) is an irreducible representation of the centralizer $C_G(\sigma)$ of some $\sigma \in \mathcal{O}$.

We denote by $M(\mathcal{O}, \rho)$ the Yetter-Drinfel'd module and by $\mathfrak{B}(\mathcal{O}, \rho)$ the associated Nichols algebra.

Conjugacy classes of $\mathbb{D}_m$ are

- $\mathcal{O}_{h^n} = \{h^n\}$, $C_{h^n} = \mathbb{D}_m$.
- $\mathcal{O}_{h^i} = \{h^{\pm i}\}$, $C_{h^i} = \langle h \rangle \simeq \mathbb{Z}/m$, Rep: $\chi_{(k)}$, $\chi_{(k)}(h) = \omega^k$.
- $\mathcal{O}_g = \{gh^j : j \text{ even}\}$, $\mathcal{O}_{gh} = \{gh^j : j \text{ odd}\}$

Recall the irreducible representations of $\mathbb{D}_m$:

- $n-1$ irred. repr. of degree 2, $\rho_\ell : \mathbb{D}_m \to \mathbf{GL}(2, \Bbbk)$,

$$\rho_\ell(g^a h^b) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}^a \begin{pmatrix} \omega^\ell & 0 \\ 0 & \omega^{-\ell} \end{pmatrix}^b, \quad 1 \leq \ell < n.$$

- 4 irred. repr. of degree 1:

σ	1	h^n	h^i, $1 \leq b \leq n-1$	g	gh
χ_1	1	1	1	1	1
χ_2	1	1	1	-1	-1
χ_3	1	$(-1)^n$	$(-1)^i$	1	-1
χ_4	1	$(-1)^n$	$(-1)^i$	-1	1

Andruskiewitsch & Fantino determined the dimension of $\mathfrak{B}(\mathcal{O}_{h^n}, \pi)$ and $\mathfrak{B}(\mathcal{O}_{h^i}, \chi_{(k)})$.

For the others we have

Lemma [FG]

$\dim \mathfrak{B}(\mathcal{O}_g, \rho) = \dim \mathfrak{B}(\mathcal{O}_{gh}, \eta) = \infty$ for all $\rho \in \widehat{C_{\mathbb{D}_m}(g)}$ and $\eta \in \widehat{C_{\mathbb{D}_m}(gh)}$.

Summarizing

Conj. class	**Centr.**	**Rep.**	$\dim \mathfrak{B}(V)$
e	$\mathbb{D}_m$	any	∞ [AF]
$\mathcal{O}_{h^n} = \{h^n\}$, $\mid \mathcal{O}_{h^n} \mid = 1$	$\mathbb{D}_m$	χ_1, χ_2, χ_3, χ_4, ρ_ℓ, ℓ even	∞ [AF]
		ρ_ℓ, ℓ odd	4 [AF] $\mathfrak{B}(M_\ell)$
$\mathcal{O}_{h^i} = \{h^{\pm i}\}$, $i \neq 0, n$, $\mid \mathcal{O}_{h^i} \mid = 2$	$\mathbb{Z}/m \simeq \langle h \rangle$	$\chi_{(k)}$, $\omega^{ik} = -1$	4 [AF] $\mathfrak{B}(M_{(i,k)})$
		$\chi_{(k)}$, $\omega^{ik} \neq -1$	∞ [AF]
$\mathcal{O}_g = \{gh^j : j \text{ even}\}$ $\mid \mathcal{O}_g \mid = n$	$\mathbb{Z}/2 \times \mathbb{Z}/2 \simeq \langle g \rangle \oplus \langle h^n \rangle$	any	∞
$\mathcal{O}_{gh} = \{gh^j : j \text{ odd}\}$ $\mid \mathcal{O}_{gh} \mid = n$	$\mathbb{Z}/2 \times \mathbb{Z}/2 \simeq \langle gh \rangle \oplus \langle h^n \rangle$	any	∞

- Define $\mathcal{I} = \{I = \coprod_{s=1}^{r}\{(i_s, k_s)\} : \omega^{i_s k_s} = -1 \text{ and } \omega^{i_s k_t + i_t k_s} = 1, 1 \leq i_s < n, 1 \leq k_s < m\}$ and

$$M_I = \bigoplus_{(i,k)\in I} M_{(i,k)}$$

Then $\mathfrak{B}(M_I) = \bigwedge M_I$ and $\dim \mathfrak{B}(M_I) = 4^{|I|}$.

- Define $\mathcal{L} = \{L = \coprod_{s=1}^{r}\{\ell_s\} : 1 \leq \ell_1, \dots, \ell_r < n, \text{ odd}\}$ and

$$M_L = \bigoplus_{\ell\in L} M_\ell.$$

Then $\mathfrak{B}(M_L) = \bigwedge M_L$ and $\dim \mathfrak{B}(M_L) = 4^{|L|}$.

- Define $\mathcal{K} = \{(I, L) : \ I \in \mathcal{I}, L \in \mathcal{L}$ and $\omega^{i\ell} = -1, k$ odd, $\forall\ (i,k) \in I, \ell \in L\}$ and

$$M_{I,L} = \left(\bigoplus_{(i,k)\in I} M_{(i,k)} \right) \oplus \left(\bigoplus_{\ell \in L} M_\ell \right).$$

Then $\mathfrak{B}(M_{I,L}) \simeq \bigwedge M_{I,L}$ and $\dim \mathfrak{B}(M_{I,L}) = 4^{|I|+|L|}$.

Theorem [FG]

Let $\mathfrak{B}(M)$ be a finite-dimensional Nichols algebra in ${}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$. Then $\mathfrak{B}(M) \simeq \bigwedge M$, with M isomorphic to M_I with $I \in \mathcal{I}$, or M_L with $L \in \mathcal{L}$, or $M_{I,L}$ with $(I, L) \in \mathcal{K}$.

Using that all finite-dimensional Nichols algebras are exterior algebras one can prove the generation in degree one:

Theorem

Let H be a finite-dimensional pointed Hopf algebra with $G(H) = \mathbb{D}_m$. Then H is generated by group-likes and skew-primitives.

i. e. $\operatorname{gr} H \simeq \mathfrak{B}(M)\#\Bbbk\mathbb{D}_m$ for some M.

Let $M \in {}^{\Bbbk\mathbb{D}_m}_{\Bbbk\mathbb{D}_m}\mathcal{YD}$. For all $1 \leq r, s < m$, let $M_r^s = \{a \in M : \ \delta(a) = h^s \otimes a, h \cdot a = \omega^r a\}$. Then $M = \bigoplus_{r,s} M_r^s$.

Using the description obtained above we find the possible deformations of the relations of the Nichols algebras over $\mathbb{D}_m$:

Proposition [FG]

Let H be a finite-dimensional pointed Hopf algebra with $G(H) = \mathbb{D}_m$ and infinitesimal braiding M. Let $a \in M_r^s, b \in M_u^v$ with $1 \leq r, s, u, v < m$ and denote $x = \sigma(a\#1)$, $y = \sigma(b\#1)$. Then there exists $\lambda \in \Bbbk^\times$ such that

$$xy + yx = \delta_{u,m-r}\lambda(1 - h^{s+v}).$$

Not all Nichols algebras admit deformations:

Lemma [FG]

Let H be a finite-dimensional such that its infinitesimal braiding M is isomorphic to M_I with $I = (i, k) \subseteq \mathcal{I}, k \neq n$ or M_L with $L \in \mathcal{L}$.Then $H \simeq \mathfrak{B}(M_I)\#\Bbbk\mathbb{D}_m$ or $H \simeq \mathfrak{B}(M_L)\#\Bbbk\mathbb{D}_m$, resp.

Using the proposition we define the quadratic algebras $A_I(\lambda, \gamma)$ and $B_{I,L}(\lambda, \gamma, \theta, \mu)$ as above and the first part of the main theorem is proved.
To prove that these algebras are liftings one first shows that

$$\dim A_I(\lambda, \gamma) \leq |\mathbb{D}_m| \dim \mathfrak{B}(M_I) \text{ and}$$
$$\dim B_{I,L}(\lambda, \gamma, \theta, \mu) \leq |\mathbb{D}_m| \dim \mathfrak{B}(M_{I,L}).$$

The equality follows by finding a representation whose restriction to $\mathbb{D}_m$ is faithful and is not trivial on the skew-primitives.

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