Twisted homogeneous racks of type D

Fernando Fantino (National University of Córdoba, Argentina)
fantino@famaf.unc.edu.ar

The problem of classifying finite-dimensional pointed Hopf algebras over non-abelian finite groups reduces in many cases to a question on conjugacy classes or, more generally, on a (twisted homogeneous) racks and a 2-cocycles. The racks of type D are a distinguished family of racks since they give arise to Nichols algebras of dimension infinite for any cocycle.

In this talk, we present some techniques to check when a twisted homogeneous rack (THR) is of type D and present a list of known THR of type D for alternating and sporadic groups.

Bibliography

[1] N. Andruskiewitsch, G. A. García, F. Fantino and L. Vendramin, *On twisted homogeneous racks of type D.* Rev. Unión Mat. Argent. **51** No. 2 (2010), 1-16.

[2] —–, *On Nichols algebras associated to simple racks.* Contemp. Math., to appear. Preprint arXiv:1006.5727.

[3] F. Fantino and L. Vendramin, *On twisted conjugacy classes of type D in sporadic groups.* In preparation.

Twisted homogeneous racks of type D

Fernando Fantino

Universidad Nacional de Córdoba &
Université Paris Diderot - Paris 7

Hopf algebras and tensor categories
Almería, July 4-8, 2011

Plan of the talk.

1. **The problem.**
2. **Main results.**
3. **Some comments.**

The problem.

Classification of finite-dimensional complex pointed Hopf algebras in the context of the *Lifting method*.

Important Step: determination of all finite-dimensional Nichols algebras of braided vector spaces arising from Yetter-Drinfled modules over groups.

Reformulation: to study finite-dimensional Nichols algebras of braided vector spaces arising from pairs (X, q), X a rack and q a 2-cocycle of X.

Racks.

A *rack* is $(X, \triangleright)$, $X \neq \emptyset$ and $\triangleright : X \times X \to X$ a map such that

- for every $x \in X$, $x \triangleright - : X \to X$ is bijective,
- $x \triangleright (y \triangleright z) = (x \triangleright y) \triangleright (x \triangleright z)$, for all x, y, $z \in X$.

Examples

- A subset X of a group G stable by conjugation of elements of G.
- A (twisted) conjugacy class of a group.

A *2-cocyle of degree n*, $n \in \mathbb{N}$, is a function $q : X \times X \to \mathrm{GL}(n, \mathbb{C})$ such that for all x, y, $z \in X$,

$$q_{x,y \triangleright z}\, q_{y,z} = q_{x \triangleright y, x \triangleright z}\, q_{x,z}.$$

Definition

$(X, \triangleright)$ is said to be of *type D* if there exists a subrack $Y = R \coprod S$ of X such that for some $r \in R$, $s \in S$,

$$r \triangleright (s \triangleright (r \triangleright s)) \neq s.$$

Properties of racks of type D.

1. If $Y \subseteq X$ is a subrack of type D, then X is of type D.
2. Let Z be a finite rack and $Z \to X$ an epimorphism. If X is of type D, then Z is of type D.

Theorem [AFGV1]
If X is of type D, then the Nichols algebra associated with (X, q) is infinite dimensional for all 2-cocycle q.
This use a result of Heckenberger-Schneider.

Definition
A rack X is said to be simple if $|X| > 1$ and it has no proper quotients.

All (indecomposable) rack has a projection onto a simple rack, i. e.

Our problem
Determine all simple racks of type D.

Finite simple racks.

A finite simple rack belongs to one of the following classes:

a. simple affine racks;
b. conjugacy classes in non-abelian finite simple groups;
c. twisted conjugacy classes (TCC) in non-abelian finite simple groups;
d. simple *twisted homogeneous racks* (THR) of class (L, t, θ): twisted conjugacy classes corresponding to (G, u), where
   - $G = L^t$, with L a non-abelian finite simple group and $t > 1$,
   - $u = u(\theta) \in \mathrm{Aut}(L^t)$ is given by

$$u(\ell_1, \dots, \ell_t) = (\theta(\ell_t), \ell_1, \dots, \ell_{t-1}), \quad \ell_1, \dots, \ell_t \in L$$

   with $\theta \in \mathrm{Aut}(L)$.

Andruskiewitsch-Graña, Joyce.

Twisted conjugacy classes (TCC).

G finite group, $x \in G$, $u \in \mathrm{Aut}(G)$. The *twisted conjugacy class of* x is:

$$\mathcal{O}_x^{G,u} := \{y \rightharpoonup_u x := y\, x\, u(y^{-1}) \,:\, y \in G\}.$$

- $\mathcal{O}_x^{G,u}$ is a rack with $y \triangleright_u z = y\, u(z\, y^{-1})$, $y, z \in \mathcal{O}_x^{G,u}$.
- TCC depend on the class of u in $\mathrm{Out}(G)$.
- A twisted conjugacy class in G is isomorphic to a conjugacy class in the group $G \rtimes \langle u \rangle$ contained in $G \times \{u\}$:

$$\mathcal{O}_{(x,u)}^{G \rtimes \langle u \rangle} = \mathcal{O}_x^{G,u} \times \{u\}.$$

Results.

Simple racks of type (a): they have very few subracks.

Simple racks as in (b) have been considered in many articles: [AFGV1] and [AFGV2] for alternating and sporadic groups, respectively; [FGV] for $\mathrm{PSL}(2, q)$.

Let $\mathbb{A}_m$ be the alternating group, $m \geq 5$.
TCC in $\mathbb{A}_m$ are conjugacy classes in $\mathbb{S}_m$ not contained in $\mathbb{A}_m$.

Theorem [AFGV1]

Let $\mathcal{O}$ be a conjugacy class of $\mathrm{Aut}(\mathbb{A}_m) \setminus \mathbb{A}_m$ different from the conjugacy class of $(1\,2)(3\,4\,5) \in \mathbb{S}_5$ and $(1\,2) \in \mathbb{S}_m$, then $\mathcal{O}$ is of type D.

TCC of type D in sporadic simple groups.

Let L be one of the following sporadic simple groups

$$M_{12},\ M_{22},\ J_2,\ J_3,\ Suz,\ HS,\ McL,\ He,\ Fi_{22},\ ON,\ Fi'_{24},\ HN,\ T.$$

It is well-known that $\mathrm{Aut}(L) \simeq L \rtimes \mathbb{Z}_2$.

Theorem [FV]

Let $\mathcal{O}$ be a conjugacy class of $\mathrm{Aut}(L) \setminus L$ not listed in the table below. Then $\mathcal{O}$ is of type D.

Group	Classes	Group	Classes
$\mathrm{Aut}(M_{22})$	**2B**	$\mathrm{Aut}(J_3)$	34A, 34B
$\mathrm{Aut}(HS)$	**2C**	$\mathrm{Aut}(ON)$	38A, 38B, 38C
$\mathrm{Aut}(Fi_{22})$	**2D**	$\mathrm{Aut}(McL)$	22A, 22B
		$\mathrm{Aut}(Fi'_{24})$	2C, 2D, 46A,46B

Corollary [FV]

If $L = M_{12}$, J_2, Suz, He, HN or T, then $\mathrm{Aut}(L)$ does not have non-trivial finite-dimensional complex pointed Hopf algebras.

Twisted homogeneous racks (THR).

L a finite group, $t \in \mathbb{N}$, $t > 1$, $\theta \in \mathrm{Aut}(L)$, $G = L^t$,

$$u(\ell_1, \dots, \ell_t) = (\theta(\ell_t), \ell_1, \dots, \ell_{t-1}), \qquad \ell_1, \dots, \ell_t \in L.$$

Denote:

- $\mathcal{C}_{(x_1,\dots,x_t)}$ = TCC of $(x_1, \dots, x_t)$ in L^t,
- $\mathcal{C}_\ell := \mathcal{C}_{(e,\dots,e,\ell)}$, $\ell \in L$.

Proposition

1. If $(x_1, \dots, x_t) \in L^t$ and $\ell = x_t x_{t-1} \cdots x_2 x_1$, then $\mathcal{C}_{(x_1,\dots,x_t)} = \mathcal{C}_\ell$.
2. $\mathcal{C}_\ell = \mathcal{C}_k$ iff $k \in \mathcal{O}_\ell^{L,\theta}$; hence

$$\mathcal{C}_\ell = \{(x_1, \dots, x_t) \in L^t : \; x_t x_{t-1} \cdots x_2 x_1 \in \mathcal{O}_\ell^{L,\theta}\}.$$

3. { TCC of L} $\longleftrightarrow$ { THR of class(L, t, θ)}, $\mathcal{O}_\ell^{L,\theta} \mapsto \mathcal{C}_\ell$.
4. $|\mathcal{C}_\ell| = |L|^{t-1} |\mathcal{O}_\ell^{L,\theta}|$.

Proposition [AFGaV1]

Let $\ell \in L^\theta$. If any of the following holds:

- ℓ is quasi-real of type j, $t \geq 3$ or $t = 2$ and $\mathrm{ord}(\ell) \nmid 2(1-j)$;
- $\mathrm{ord}(\ell)$ even and $t \geq 6$ even;
- ℓ involution, t odd and $\mathcal{O}_\ell^{L^\theta}$ of type D;
- $t = 4$ and there exists $x \in C_{L^\theta}(\ell)$ with $\mathrm{ord}(x) = 2m > 2$, $m \in \mathbb{N}$;
- $t = 2$ and there exists $x \in C_{L^\theta}(\ell)$ with $\mathrm{ord}(x) = 2m > 4$, $m \in \mathbb{N}$;
- ℓ involution, $t = 2$, and there exists $\psi : \mathbb{D}_n \to L^\theta$ a group monomorphism, with $n \geq 3$ and $\ell = \psi(x)$ for some $x \in \mathbb{D}_n$ involution;
- $\ell = e$ and $(t, |L^\theta|)$ is divisible by an odd prime p;
- $\ell = e$ and $(t, |L^\theta|)$ is divisible by $p = 2$ and $t \geq 6$;
- $\ell = e$, $t = 4$ and there exists $x \in L^\theta$ with $\mathrm{ord}(x) = 2m > 2$, $m \in \mathbb{N}$;
- $\ell = e$, $t = 2$ and there exists $x \in L^\theta$ with $\mathrm{ord}(x) = 2m > 4$, $m \in \mathbb{N}$;

then $\mathcal{C}_\ell$ is of type D.

THR of type D in alternating groups.

Theorem [AFGaV1]

Let L be $\mathbb{A}_m$, $m \geq 5$, $\theta \in \mathrm{Aut}(L)$, $t \geq 2$ and $\ell \in L$. If $\mathcal{C}_\ell$ is a THR of class (L, t, θ) not listed in two tables below, then $\mathcal{C}_\ell$ is of type D.

Table: THR $\mathcal{C}_\ell$ of type $(\mathbb{A}_m, t, \theta)$, $\theta = \mathrm{id}$, $t \geq 2$, $m \geq 5$, not known of type D.

n	ℓ	Type of ℓ	t
any	e	(1^n)	odd, $(t, n!) = 1$
5		(1^5)	$\mathbf{2}$, 4
6		$(\mathbf{1^6})$	$\mathbf{2}$
5	involution	$(1, 2^2)$	4, odd
6		$(1^2, 2^2)$	odd
8		(2^4)	odd
any	order 4	$(1^{r_1}, 2^{r_2}, 4^{r_4})$, $r_4 > 0$, $r_2 + r_4$ even	2

THR of type D in alternating groups.

Table: THR $\mathcal{C}_\ell$ of type $(\mathbb{A}_n, t, \theta)$, $\theta = \iota_{(1\,2)}$, $t \geq 2$, $n \geq 5$, not known of type D.

n	Type of $\ell(1\ 2)$	t
any	$(1^{s_1}, 2^{s_2}, \dots, n^{s_n})$, $s_1 \leq 1$ and $s_2 = 0$ $s_h \geq 1$, for some h, $3 \leq h \leq n$	any
	$(1^{s_1}, 2^{s_2}, 4^{s_4})$, $s_1 \leq 2$ or $s_2 \geq 1$, $s_2 + s_4$ odd, $s_4 \geq 1$	2
5	$(1^3, 2)$	2, 4
6	$(1^4, 2)$	2
	(2^3)	2
7	$(1, 2^3)$	2, odd
8	$(1^2, 2^3)$	odd
10	(2^5)	odd

THR of type D in sporadic simple groups.

Difficult task.

Theorem [AFGaV1]

Let L be a sporadic group, $\theta = \mathrm{id}$, $t \geq 2$ and $\ell \in L$. If $\mathcal{C}_\ell$ is a THR of class (L, t, θ) not listed in the table below, then $\mathcal{C}_\ell$ is of type D.

Table: THR $\mathcal{C}_\ell$ of type (L, t, θ), L sporadic group, $\theta = \mathrm{id}$, not known of type D.

sporadic group	Type of ℓ or class name of $\mathcal{O}_\ell^L$	t
any	1A	$(t, \lvert L\rvert) = 1$, t odd
	$\mathrm{ord}(\ell) = 4$	2
T, J_2, Fi_{22}, Fi_{23}, Co_2	2A	odd
B	2A, 2C	odd
Suz	6B, 6C	any

Thank you.

References.

- N. Andruskiewitsch, F. Fantino, G. A. García and L. Vendramin, *On twisted homogeneous racks of type D*, Rev. Un. Mat. Argentina **51** (2010),no. 2, 1–16.
- ____, *On Nichols algebras associated to simple racks*, Groups, Algebras and Applications, Contemp. Math. **537**, Amer. Math. Soc., Providence, RI, 2011, 31–56.
- F. Fantino and L. Vendramin, *On twisted conjugacy classes of type D in sporadic simple groups*, submitted. Preprint `arXiv:1107.0310`.

Other references.

N. Andruskiewitsch, F. Fantino, M. Graña and L. Vendramin, *Finite-dimensional pointed Hopf algebras with alternating groups are trivial*, Ann. Mat. Pura Appl. (4) **190** (2011), no. 2, 225–245.

____, *Pointed Hopf algebras over the sporadic simple groups*, J. Algebra **325** (2011),no. 2, 305–320.

N. Andruskiewitsch and M. Graña, *From racks to pointed Hopf algebras*, Adv. Math. **178** (2003),no. 2, 177–243.

N. Andruskiewitsch, I. Heckenberger and H.-J. Schneider, *The Nichols algebra of a semisimple Yetter-Drinfeld module*, Amer. J. Math. **132** (2010),no. 6, 1493–1547.

____, *On Nichols algebras over* $\mathrm{PSL}(2, q)$ *and* $\mathrm{PGL}(2, q)$, J. Algebra Appl. **9** (2010),no. 2, 195–208.

I. Heckenberger and H.-J. Schneider, *Root systems and Weyl groupoids for Nichols algebras*, Proc. London Math. Soc. **101** (2010),no. 3, 623–654