Classifying Hopf algebras of a given dimension

Margaret Beattie (Mount Allison University, Canada)
mbeattie@mta.ca

Over an algebraically closed field, the problem of classifying all Hopf algebras even for some given small dimension, such as 16 or 32, or for a class of dimensions, such as p, pq , pq^2, etc, for p, q prime, is a difficult one. Some recent techniques using the coradical filtration are due to D. Fukuda; he applied these to dimensions 18 and 30. Cheng and Ng have recently investigated Hopf algebras of dimension p in the Yetter-Drinfeld category over the 4-dimensional Sweedler Hopf algebra and used these results to study dimension $4p$. They show that Hopf algebras of dimensions $20, 28$, or 44 are either semisimple, pointed or copointed.

In this talk some more techniques will be mentioned with applications to dimension p^3 in mind. Hopf algebras of dimension 27 will be completely described.

This is joint work with G.A. García.

Classifying Hopf algebras of a given dimension

Margaret Beattie [1]

Mt Allison U.

July 7, 2011

[1] Joint work with Gastón García, U. Cordoba.

The problem

The question of classification of all Hopf algebras of a given dimension over an algebraically closed field k of characteristic zero up to isomorphism dates back 35 years or more to Kaplansky's monograph on bialgebras. A little progress has been made....

- 1994 Kac-Zhu Theorem: Hopf algebras of prime dimension are group algebras.
- 1995, 1996 , Masuoka: Semisimple Hopf algebras of dim p^2 are group algebras. Classified all of dimension p^3, not all are group algebras.
- 1998 Andruskiewitsch & Schneider, Caenepeel & Dăscălescu, Ştefan & van Oystaeyen independently: Found $(p-1)(p+9)/2$ isomorphism classes of pointed Hopf algebras of dim p^3. Two isomorphism types have nonpointed duals.

very brief history

- 1998 - 2004 Masuoka, Izumi & Kasaki, Sommerhaüser, Gelaki, Westreich, Etingof : Semisimple Hopf algebras of dimension pq are group algebras or their duals.
- 1999 -2001 Natale: classification of semisimple Hopf algebras of dimension pq^2, also Masuoka
- 2001 Andruskiewitsch & Natale: pointed Hopf algebras of dim pq^2.
- 2002 Ng: Nonsemisimple Hopf algebras of dimension p^2 are Taft Hopf algebras.
- 2004 Natale: Semisimple Hopf algebra of dimension pq^r

history cont'd

- 2005 G.A.García: Classified the ribbon Hopf algebras of dimension p^3 and showed that a nonsemisimple nonpointed noncopointed Hopf algebra of dimension p^3 is of type (p, p) or $(p, 1)$ and has no nontrivial normal sub-Hopf algebra.
- 2008 Ng: Hopf algebras of dim pq with $2 < p < q \leq 4p + 11$ are semisimple. Also earlier results of Ng (dim $2p$ done in general), as well as Etingof & Gelaki, D. Fukuda.
- 2009 Hilgemann & Ng: All Hopf algebras of dimension $2p^2$ are semisimple, pointed or copointed
- 2011 Cheng & Ng: partial classification of Hopf algebras of dimension $4p$; dimensions $20, 28, 44$ are semisimple, pointed or copointed.

Methods

No standard methods of attack – but some approaches are:

(i) S. Natale: Hopf algebras H generated by a simple subcoalgebra of dimension 4 stable under the antipode have a central exact sequence:

$$k^G \hookrightarrow H \twoheadrightarrow A$$

where G is a finite group and A is copointed nonsemisimple.

(ii) H nonsemisimple, then the trace of the square of the antipode is 0.

(iii) Study Hopf algebras in categories of Yetter-Drinfeld modules.

(iv) Projective covers of simple modules

(v) Injective envelopes of simple comodules

(vi) Dimension arguments - useful for some general and for some particular small dimensions.

Some recent generalizations

Natale's theorem about Hopf algebras generated by a simple subcoalgebra of dimension 4 stable under the antipode depends on a theorem of D. Ştefan about matrix coalgebras. Here is part of the generalization due to C. Vay:

Proposition

Let $D \cong \mathcal{M}^*(d,k)$. If f is a coalgebra automorphism of D of finite order n, there is a comatrix basis (also called a multiplcative matrix) $\mathbf{e}$ *for D consisting of eigenvectors for f and such that $f(e_{ij}) = \omega_i \omega_j^{-1} e_{ij}$ for some scalars ω_i.*

Other generalizations of methods in the literature use a description of the coradical filtration due to Nichols and explained in [AN] 2001, together with some results of D. Fukuda.

On the coradical filtration

For C a coalgebra, there is a (non-unique) coalgebra projection π from C to its coradical C_0 with kernel I. Define:

$$\rho_L := (\pi \otimes C)\Delta : C \to C_0 \otimes C \qquad \text{and} \qquad \rho_R := (C \otimes \pi)\Delta : C \to C \otimes C_0$$

Define a sequence of subspaces P_n recursively by

$$\begin{aligned} P_0 &= 0; \\ P_1 &= \{x \in C : \ \Delta(x) = \rho_L(x) + \rho_R(x)\} = \Delta^{-1}(C_0 \otimes I + I \otimes C_0), \\ P_n &= \{x \in C : \ \Delta(x) - \rho_L(x) - \rho_R(x) \in \sum_{1 \le i \le n-1} P_i \otimes P_{n-i}\}, \quad n \ge 2. \end{aligned}$$

$P_n = C_n \cap I$ for $n \geq 0$ and $C_n = C_0 \oplus P_n$. Also $H_n = H_0 \oplus P_n$ for all $n \geq 0$. H_n and P_n are H_0-sub-bicomodules of H via ρ_R and ρ_L.

Coradical filtration

Let $C_0 = \bigoplus_{\tau \in I} C_\tau$, where $C_\tau \cong M^c(d_\tau, k)$.

Every simple C_0-bicomodule has coefficient coalgebras C_τ, C_γ and has dimension $d_\tau d_\gamma = \sqrt{\dim C_\tau \dim C_\gamma}$ for some $\tau, \gamma \in I$.

Some notation:

- $P_n^{\tau,\gamma}$ is the isotypic component of the H_0-bicomodule P_n of type the simple bicomodule with coalgebra of coefficients $C_\tau \otimes C_\gamma$.
- $P^{\tau,\gamma} = \sum_n P_n^{\tau,\gamma}$.
- If g is grouplike, write $P_n^{g,\tau}$ for $P_n^{kg,\tau}$.
- Similar abbreviations for SC_τ, gC_τ, etc.
- If $\Gamma \subseteq G(C)$, write $P^{\Gamma,\Gamma}$ for $\sum_{g,h \in \Gamma} P^{g,h}$.

Dimension arguments

The subspace $P_n^{\tau,\gamma}$ is called *non-degenerate* if $P_n^{\tau,\gamma} \not\subseteq P_{n-1}$. Then if H is a Hopf algebra:

$$\dim P_n^{\tau,\gamma} = \dim P_n^{S\gamma,S\tau} = \dim P_n^{g\tau,g\gamma} = \dim P_n^{\tau g,\gamma g},$$

Some standard useful facts are:

- For all n, $|G(H)|$ divides the dimension of H_n, P_n, and $H_{0,d}$ where this last is the sum of the simple subcoalgebras of dimension d^2. [AN]
- If H is noncosemisimple with no nontrivial skew-primitives, then for all $g \in G(H)$ there is a simple subcoalgebra C of H of dimension greater than 1 such that $P_1^{g,C}$ is nonzero. [BD]

results of Fukuda

Lemma

If the subspace $P_n^{\tau,\gamma}$ is nondegenerate for some $n > 1$ then for all $1 \leq i \leq n-1$ there is a simple coalgebra C_i such that $P_i^{\tau,C_i}, P_{n-i}^{C_i,\gamma}$ are nondegenerate.

Lemma

Let C, D be simple subcoalgebras such that $P_m^{C,D}$ is nondegenerate. If either

$$\dim C \neq \dim D \quad \text{or} \quad \dim P_m^{C,D} - \dim P_{m-1}^{C,D} \neq \dim C$$

there exists a simple subcoalgebra E and $t \geq m+1$ such that $P_t^{C,E}$ is nondegenerate.

Bound on the dimension

Fukuda's results lead to an improved bound on the dimension of Hopf algebras with no nontrivial skew-primitive elements from an old result of B/Dăscălescu:

Proposition

Let H be a non-cosemisimple Hopf algebra with no nontrivial skew-primitives. Then

$$\dim(H) \geq \dim(H_0) + (2n+1)|G| + n^2,$$

where n^2 is the dimension of the smallest simple subcoalgebra of H of dimension greater than 1.

Application: Every Hopf algebra of dimension 27 and grouplikes of order 3 has a nontrivial skew-primitive element.

Dim 27 with only trivial grouplike elements

Suppose H has dimension 27 and $|G(H)| = 1$. The possible coradicals for H are $k \cdot 1 \oplus E$ where E is:

- $\mathcal{M}^*(2,k)^n$ with $n = 1,2,3,4,5,6$;
- $\mathcal{M}^*(3,k)^n$ with $n = 1,2$;
- $\mathcal{M}^*(4,k)$;
- $\mathcal{M}^*(5,k)$;
- $\mathcal{M}^*(2,k)^n \oplus \mathcal{M}^*(3,k)^m$ with $(n,m) = (1,1),(2,1),(3,1),(4,1),(1,2)$;
- $\mathcal{M}^*(2,k)^n \oplus \mathcal{M}^*(3,k)^m \oplus \mathcal{M}^*(4,k)$ with $0 \leq n,m$ and $0 < 4n + 9m + 16 < 26$.

A simple application of the proposition bounding the dimension eliminates many of these....

Dim 27 with only trivial grouplike elements

What remains:

- $\mathcal{M}^*(2,k)^n$ with $n = 1, 2, 3, 4, 5, 6$;
- $\mathcal{M}^*(3,k)^n$ with $n = 1, 2$;
- $\mathcal{M}^*(4,k)$;
- $\mathcal{M}^*(5,k)$;
- $\mathcal{M}^*(2,k)^n \oplus \mathcal{M}^*(3,k)^m$ with $(n,m) = (1,1), (2,1), (3,1), (4,1), (1,2)$;
- $\mathcal{M}^*(2,k)^n \oplus \mathcal{M}^*(3,k)^m \oplus \mathcal{M}^*(4,k)$ with $0 \leq n, m$ and $0 < 4n + 9m + 16 < 26$.

Now recall by Natale's theorem that here if H is generated by a simple subcoalgebra of dimension 4 stable under the antipode, then H is copointed.

Dim 27 with only trivial grouplike elements

What remains:

- $\mathcal{M}^*(2,k)^n$ with $n = 1, 2, 3, 4, 5, 6$;
- $\mathcal{M}^*(3,k)^n$ with $n = 1, 2$;
- $\mathcal{M}^*(4,k)$;
- $\mathcal{M}^*(5,k)$;
- $\mathcal{M}^*(2,k)^n \oplus \mathcal{M}^*(3,k)^m$ with $(n,m) = (1,1), (2,1), (3,1), (4,1), (1,2)$;
- $\mathcal{M}^*(2,k)^n \oplus \mathcal{M}^*(3,k)^m \oplus \mathcal{M}^*(4,k)$ with $0 \leq n, m$ and $0 < 4n + 9m + 16 < 26$.

Now with exactly two copies of $\mathcal{M}^*(2,k)$ in the coradical either S fixes both or permutes them properly. The first case is impossible by Natale's theorem and the second is impossible by a counting result obtained from Fukuda's lemmas.

Dimension lemma

Lemma

Let H be a Hopf algebra of dimension 27. Then the coradical $H_0 \ncong k \cdot 1 \oplus C \oplus D$ where $S(C) = D$ and $C \cong D \cong \mathcal{M}^*(2, k)$.

Idea of the proof:

$$27 = \dim(H_0) + \dim(P^{1,1}) + \sum_{E,F \in \{C,D\}} [\dim(P^{1,E}) + \dim(P^{E,1}) + \dim(P^{E,F})]$$

so the dimension of $P^{1,1}$ is congruent to 2 mod 4. From Fukuda, $P_3^{1,1}$ is nondegenerate, so that again by Fukuda, $P_2^{1,E}$ is nondegenerate for some $E \in \{C, D\}$. Then count the dimensions.

Dim 27 with only trivial grouplike elements

What remains:

- $\mathcal{M}^*(2,k)^n$with $n = 1, 2, 3, 4, 5, 6$;
- $\mathcal{M}^*(3,k)^n$ with $n = 1, 2$;
- $\mathcal{M}^*(4,k)$;
- $\mathcal{M}^*(5,k)$;
- $\mathcal{M}^*(2,k)^n \oplus \mathcal{M}^*(3,k)^m$ with $(n,m) = (1,1), (2,1), (3,1), (4,1), (1,2)$;
- $\mathcal{M}^*(2,k)^n \oplus \mathcal{M}^*(3,k)^m \oplus \mathcal{M}^*(4,k)$ with $0 \leq n, m$ and $0 < 4n + 9m + 16 < 26$.

The following lemma builds on the results of Fukuda:

Lemma

If dim (H) is divisible by $N > 2$, $H_0 \cong k \cdot 1 \oplus E$ where E is a sum of simple subcoalgebras each of dimension divisible by N^2 then the dimension of H is greater than or equal to dim$(E) + 5N + 2N^2$.

And this eliminates the last possibility.

Dimension 27 type $(3, 3)$

Recall that:

- If dim (H) is 27 and $G(H) \cong C_3$ then H has a nontrivial skew-primitive element.
- p divides the dimension of each of P_n and each $H_{0,d}$
- The only possible coradicals for H of dimension 27 with $G(H) \cong C_3$ are
  - $kC_3 \oplus \mathcal{M}^*(2,k)^3$
  - $kC_3 \oplus \mathcal{M}^*(2,k)^3 \oplus \mathcal{M}^*(3,k)$ - impossible by counting dimensions
  - $kC_3 \oplus \mathcal{M}^*(3,k)$ or
  - $kC_3 \oplus \mathcal{M}^*(3,k)^2$.

Some general results for Hopf algebras of dimension p^3 and type (p, p) will eliminate each of these possibilities.

Dimension p^3, type (p,p), coradical with copies of $\mathcal{M}^*(p-1,k)$

Assume H is of dim p^3, type (p,p) nonsemisimple, nonpointed, non-copointed.

Proposition

Suppose H contains a Taft Hopf algebra T and $T = H^{G(H),G(H)}$. Suppose that $d^2 < p^2$ divides the dimension of every simple subcoalgebra of H of dimension greater than 1. Then

- *d must divide $p-1$, and if d is even then $2d$ divides $p-1$.*
- *Also if $P_1^{g,E} = 0$ for all $g \in G(H)$ and E a simple subcoalgebra of H of dimension greater than 1 then d^2 divides $p-1$.*

Corollary

H of dimension p^3. Then $H_0 \ncong kC_p \oplus \mathcal{M}^*(p-1,k)^p \oplus E$ where E is zero or a sum of simple coalgebras isomorphic to $\mathcal{M}^*(p,k)$.

Dimension p^3, type (p,p), coradical with copies of $\mathcal{M}^*(p,k)$

Proposition

Suppose that both H and H^* have nontrivial skew-primitive elements so that there is a Hopf algebra projection $\pi : H \to T_q$ where T_q is a Taft Hopf algebra of dimension p^2. Then H has no simple subcoalgebra D of dimension p^2 such that $P^{D,D} = 0$. If H has dimension 27 then the condition that $P^{D,D} = 0$ is not necessary.

Remark that the proof relies heavily on the fact that we have a basis e_{ij} for D consisting of eigenvectors for the inner action by a nontrivial grouplike.

Other examples

Other applications:

Corollary

Let $p > 5$ and H be of type (p, p) with $H_0 \cong kC_p \oplus \mathcal{M}^*(p,k)^t$ where $t \geq p-3$. Then H^* has no nontrivial skew primitive element.

Proof.

H has a nontrivial skew-primitive by the usual dimension arguments. Suppose that H^* also has a nontrivial skew-primitive. If $P^{D,D} \neq 0$ for all D simple of dimension p^2 then the dimension of H is at least $p^2 + 2tp^2 = (1+2t)p^2 \geq (1+2(p-3))p^2 = 2p^3 - 5p^2 > p^3$. □

Remark

The same argument shows that if $p = 5$ and $H_0 \cong kC_5 \oplus \mathcal{M}^*(5,k)^t$ with $t = 3, 4$, then H^* has no nontrivial skew-primitive.

Dim 27 Hopf algebras are semisimple, pointed or copointed

Semisimple Hopf algebras [Mas]:

(*a*) 5 of form kG, 2 of form kG^* G nonabelian.

(*c*) 4 self-dual extensions of $k[C_3 \times C_3]$ by kC_3, noncomm, noncocomm.

List of pointed and copointed Hopf algebras [AS], [CD], [SvO]. Here g denotes a grouplike, x skew-primitive, $q^3 = 1$, $\xi^9 = 1$.

(*c*) $T_q \otimes kC_3$.

(*d*) $\widetilde{T_q} := k\langle g,\ x|\ gxg^{-1} = \xi x,\ g^9 = 1,\ x^3 = 0\rangle$, x , $(g^3, 1)$-prim.

(*e*) $\widehat{T_q} := k\langle g,\ x|\ gxg^{-1} = qx,\ g^9 = 1,\ x^3 = 0\rangle$, x $(g, 1)$-prim.

(*f*) $\mathbf{r}(q) := k\langle g,\ x|gxg^{-1} = qx, g^9 = 1,\ x^3 = 1 - g^3\rangle$,$x$ $(g, 1)$-prim.

(*g*) Frobenius-Lusztig kernel $\mathbf{u}_q(\mathfrak{sl}_2)$.

(*h*) The book Hopf algebra $\mathbf{h}(q, m)$.

(*i*) Dual of the Frobenius-Lusztig kernel, $\mathbf{u}_q(\mathfrak{sl}_2)^*$. Not pointed.

(*j*) Dual of the case f), $\mathbf{r}(q)^*$. Not pointed.

Small dimensions

All Hopf algebras of dimension less than 32 have been shown to be semisimple, pointed or copointed except for dimension $24 = 2^3 \cdot 3$.

- Dim p: $2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31$
- Dim p^2: $4, 9, 25$
- Dim pq: $6, 10, 14, 15, 21, 22, 26$
- Dim p^3: 8 by Williams (also [Ş]), 27
- Dim p^4: 16 by García and Vay (Also [K], [CDR], [B].)
- Dim $2p^2$: 18 by D. Fukuda (gen'l result by Hilgemann & Ng)
- Dim $4p$: 12 Natale (N. Fukuda ss case, also Natale pq^2 ss in gen'l, Masuoka, [AN] pted), $20, 28$ Cheng and Ng
- Dim pqr: 30 by D. Fukuda

After that the next dimensions which have not been classified are $32 = 2^5$,$40 = 2^3 \cdot 5$, $42 = 2 \cdot 3 \cdot 7$, $45 = 3^2 \cdot 5$, $48 = 2^4 \cdot 3$.