Twisted partial Hopf actions

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The notion of a twisted partial Hopf action is a natural generalization of both, twisted partial group actions and partial Hopf actions. The twisted partial group actions arise in the context of graded algebras, allowing them to be classified as crossed products. The partial actions and coactions of Hopf algebras were originally used to put partial Galois extensions of commutative algebras in a broader context of Galois Corings. In this work, we define a twisted partial action of a Hopf algebra on a unital algebra, construct partial crossed products and relate them with partially cleft extensions. The globalization theorem for twisted partial Hopf actions is also discussed.

Twisted Partial Actions of Hopf Actions

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Hopf algebras and tensor categories - Almeria - Spain,
July 4-8 2011

Outline

Motivations

Why partial actions became important?

- Partial actions were first conceived to classify C^*-algebras generated by partial isometries.
- Many ordinary differential equations do not have the flux globally defined (partial action of the additive group $\mathbb{R}$.
- The situation above appears naturally in the study of geodesically incomplete Riemannian manifolds.
- Partial actions seems to be an appropriate language to describe quasicrystals and aperiodic systems.

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Partial Actions of Groups

Definition

A partial action of a group G on a unital k-algebra A, is a pair

$$(\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G})$$

where for each $g \in G$ D_g is an ideal of A generated by a central idempotent 1_g and $\alpha_g : D_{g^{-1}} \to D_g$ is a unital isomorphism (by unital we mean $\alpha_g(1_{g^{-1}}) = 1_g$) such that

(i) $D_e = A$, *and* $\alpha_e = Id_A$.

(ii) $\alpha_g(D_{g^{-1}} \cap D_h) = D_g \cap D_{gh}$.

(iii) *For all* $x \in \alpha_{h^{-1}}(D_h \cap D_{g^{-1}})$, *we have* $\alpha_g \circ \alpha_h = \alpha_{gh}$.

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Partial Actions of Groups, part 2, Example

A paradigmatic example of a partial action of a group on a unital algebra can be given by considering $\beta : G \to \mathrm{Aut}(B)$ an action of G by automorphisms of an algebra B and an ideal $A = 1_A B$ (where 1_A is a central idempotent in B)

In this case, for each $g \in G$ take
$D_g = A \cap \beta_g(A)$, and $\alpha_g = \beta_g|_{D_g}$.
It is easy to verify that $D_g = 1_g A$, where $1_g = \alpha_g(1_A)$, and that these D_g and α_g perform a partial action of G on A.

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Partial Hopf Actions

And what about partial actions of Hopf algebras?

- Were first introduced by Caenepeel and Janssen in order to put Galois Theory for partial actions into a brosder context.
- Exhibit nice properties of globalization.
- Extend some results on duality of actions.
- A tool to describe properties of ideals of G graduated algebras, (partially G-graduated algebras).

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Partial Actions of Hopf Algebras

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(Caenepeel, Janssen 08) A partial action of H on A is a linear mapping $h \otimes a \mapsto h \cdot a$ such that:

(i) $h \cdot (ab) = \sum (h_{(1)} \cdot a)(h_{(2)} \cdot b)$,

(ii) $1 \cdot a = a$,

(iii) $h \cdot (k \cdot a) = \sum (h_{(1)} \cdot \mathbf{1}_A)((h_{(2)} k) \cdot a)$.

- When H acts partially on A, we say that A is a partial H-module algebra.
- If $h \cdot \mathbf{1}_A = \epsilon(h) \mathbf{1}_A$ for all h then A turns out to be a, H-module algebra.

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Partial Actions of Hopf Algebras, Example 1

the first example of this kind of action is based on a partial action of a group G on a unital algebra A.

Define for all $a \in A$,

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Then we have

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Another example of a partial Hopf action is given by:

Induced partial Hopf actions

(Alves, B. 2010) Let B be a H-module algebra, A a right unital ideal of B, with unity $\mathbf{1}_A$. Then

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A nontrivial example: $(kG)^*$-actions

If H is a finite-dimensional Hopf algebra then H^* is a Hopf algebra as well, and H^* acts on H by

$$h^* \rightharpoonup h = \sum h^*(h_{(2)})h_{(1)}$$

Let G be a finite group and N be a normal subgroup of G. Then

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$(kG)^*$-actions, continued

Let $A = e_N kG$, an unital ideal in kG, let $\beta^* = \{p_g ; g \in G\}$ be the dual basis of $(kG)^*$.

The induced partial action on A is

$$p_g \cdot e_N x = e_N(p_g \rightharpoonup e_N x) = \begin{cases} (1/|N|)e_N x & \text{if } gx^{-1} \in N \\ 0 & \text{otherwise} \end{cases}$$

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Prelude to Twisted Partial Hopf Actions

In this talk, we are going to do a step further: To present the notion of twisted partial Hopf actions, which generalize at once:

- Twisted partial group actions.
- Partial Hopf actions.
- Hopf actions twisted by a cocycle.

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Twisted Partial Group Actions

A twisted partial action of a group G over a unital k-algebra A is a triple

$$\left(\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G}\right)$$

where D_g is an ideal of A generated by a central idempotent 1_g of A, $\alpha_g : D_{g^{-1}} \to D_g$ is an isomorphism of unital k-algebras, and for each $(g,h) \in G \times G$, $w_{g,h}$ is an invertible element in $D_g D_{gh}$, satisfying:

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(iv) $\alpha_g(\alpha_h(a)) = w_{g,h}\alpha_{gh}(a)w_{g,h}^{-1}$ for all $a \in D_{h^{-1}} D_{h^{-1}g^{-1}}$.

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Partial Crossed Products, the Group Case

Given a twisted partial action, we can construct the crossed product

$$A \rtimes_{\alpha,w} G \cong \oplus_{g\in G} D_g = \{\sum_{g\in G} a_g\delta_g | a_g \in D_g\}$$

with the product given by

$$(a_g\delta_g)(b_h\delta_h) = \alpha_g(\alpha_{g^{-1}}(a_g)b_h)w_{g,h}\delta_{gh}$$

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What is the motivation for such construction?

The caracterization of group graded algebras as crossed products by twisted partial group actions.

More specifically, given a group G and a G-graded algebra $\mathcal{B} = \oplus_{g\in G}\mathcal{B}_g$ satisfying

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it is possible to define a twisted partial action of G on $\mathcal{B}_e$, where $e \in G$ is the neutral element of the group G, such that $\mathcal{B} \cong \mathcal{B}_e \rtimes_{\alpha,\omega} G$.

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The pair (α, ω) is called a twisted partial action of H on A if the following conditions hold:

(i) $1_H \cdot a = a$.

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for all $a, b \in A$ and $h, l \in H$.

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Twisted Partial Hopf Actions, Continued

For the moment, we are not requiring that $\omega \in \mathrm{Hom}_k(H \otimes H, A)$, be neither normalized, nor invertible by convolution.

Some Special Cases

When

$$\omega(h,k) = h \cdot (k \cdot \mathbf{1}_A) = (h_{(1)} \cdot \mathbf{1}_A)(h_{(2)}k \cdot \mathbf{1}_A)$$

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A First Example: Twisted Partial Group Actions

Consider

$$\left(\{D_g\}_{g\in G}, \{\alpha_g\}_{g\in G}, \{w_{g,h}\}_{(g,h)\in G\times G}\right)$$

a twisted partial action of a group G on a unital algebra A.

Let $\alpha : kG \otimes A \to A$ and $\omega : kG \otimes kG \to A$ be the k-linear maps given respectively by

$$\alpha(g \otimes a) = \alpha_g(a1_{g^{-1}}), \quad \text{and} \quad \omega(g, h) = w_{g,h},$$

for all $a \in A$ and $g, h \in G$.

This pair (α, ω) is a twisted partial action of kG on A.

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A first Example: Twisted Partial Group Actions, Continued

Note that the condition

$$\omega(h,l) = \sum (h_{(1)} \cdot 1_A)\omega(h_{(2)}, l_{(1)})(h_{(3)} l_{(2)} \cdot 1_A)$$

is quite natural, for it reduces, in the case of groups, to

$$w_{g,h} = 1_g w_{g,h} 1_{gh},$$

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Induced Twisted Partial Action

Let B be a K-algebra measured by an action $\beta : H \otimes B \to B$, denoted by $\beta(h, b) = h \triangleright b$, which is twisted by a map $u : H \otimes H \to B$, that is,

1. $h \triangleright (ab) = \sum (h_1 \triangleright a)(h_2 \triangleright b)$.
2. $h \triangleright 1 = \epsilon(h)1$.
3. $\sum (h_{(1)} \triangleright (k_{(1)} \triangleright a))u(h_{(2)}, k_{(2)}) = \sum u(h_{(1)}, k_{(1)})(h_{(2)}k_{(2)} \triangleright a)$.

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Induced Twisted Partial Action, Part 2

Suppose now that $\mathbf{1}_A$ is a non-trivial central idempotent of B, and let A be the ideal generated by $\mathbf{1}_A$. Given $a \in A, h \in H$, define a map $\cdot : H \otimes A \to A$ as

$$h \cdot a = \mathbf{1}_A (h \triangleright a)$$

Induced Twisted Partial Action, Part 3

It follows from the definition above and the properties of the twisted global action that

$$\sum (h_{(1)} \cdot (k_{(1)} \cdot \mathbf{1}_A)) u(h_{(2)}, k_{(2)}) = \sum (h_{(1)} \cdot \mathbf{1}_A) u(h_{(2)}, k_{(1)}) (h_{(3)} k_{(2)} \cdot \mathbf{1}_A)$$

This identity motivates us to define ω by

$$\omega(h, k) = \sum (h_{(1)} \cdot \mathbf{1}_A) u(h_{(2)}, k_{(1)}) (h_{(3)} k_{(2)} \cdot \mathbf{1}_A)$$

With these definitions, one can show that $(\cdot, \omega)$ indeed define a twisted partial action of H on the algebra A.

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Partial Crossed Product

Given any twisted partial action of a Hopf algebra H on a unital algebra A, we can define on the $A \otimes H$ a product, given by

$$(a \otimes h)(b \otimes l) = \sum a(h_{(1)} \cdot b)\omega(h_{(2)}, l_{(1)}) \otimes h_{(3)} l_{(2)},$$

for all $a, b \in A$ and $h, l \in H$.

Partial Crossed Product, Part 2

Let now denote by $A\#_\omega H = (A \otimes H)(\mathbf{1}_A \otimes 1_H)$, which corresponds, to the k-submodule of$A \otimes H$ generated by the elements of the form

$$a\#h := \sum a(h_{(1)} \cdot 1_A) \otimes h_{(2)},$$

for all $a \in A$ and $h \in H$.

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Partial Crossed Product, Part 3

Theorem

Let H be a Hopf algebra with a twisted partial Hopf action on A.

(i) $\mathbf{1}_A \# 1_H$ *is the unity of* $A \#_\omega H$ *if and only if, for all* $h \in H$, *we have* $\omega(h, 1_H) = \omega(1_H, h) = h \cdot 1_A$.

(ii) *Suppose that* $\omega(h, 1_H) = h \cdot 1_A$, *for all* $h \in H$. *Then* $A \#_\omega H$ *is associative if and only if the condition*

$$\sum (h_{(1)} \cdot (l_{(1)} \cdot a))\omega(h_{(2)}, l_{(2)}) = \sum \omega(h_{(1)}, l_{(1)})(h_{(2)} l_{(2)} \cdot a) \tag{1}$$

holds and , for all $h, l, m \in H$,

$$\sum (h_{(1)} \cdot \omega(l_{(1)}, m_{(1)}))\omega(h_{(2)}, l_{(2)} m_{(2)}) = \\ = \sum \omega(h_{(1)}, l_{(1)})\omega(h_{(2)} l_{(2)}, m). \tag{2}$$

Partial Crossed Product, Part 4

When $H = kG$, the crossed product $A\#_\omega kG$ coincides with the crossed product defined for twisted partial group actions: $A \rtimes_{\alpha,w} G$.

Symmetric Twisted Partial Hopf Actions

Consider a twisted partial action of H on A. It is easy to see that $f_1(h,k) = (h \cdot \mathbf{1}_A)\epsilon(k)$ and $f_2(h,k) = (hk \cdot \mathbf{1}_A)$ are both idempotents in the convolution algebra $\mathrm{Hom}(H \otimes H, A)$. We also have that $e(h) = (h \cdot \mathbf{1}_A)$ is an idempotent in $\mathrm{Hom}(H, A)$ (and $f_1(h,k) = e(h)\epsilon(k)$).
Let us assume that both f_1 and f_2 are central in $\mathrm{Hom}(H \otimes H, A)$. In this case, from the definition of twisted partial action, one can say that ω is an element of the unital ideal $\langle f_1 * f_2 \rangle \subset \mathrm{Hom}(H \otimes H, A)$.

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Symmetric Twisted Partial Hopf Actions, II

Definition

Let $A = (A, \cdot, \omega)$ be a twisted partial H-module algebra. We will say that the partial action is symmetric if

(i) f_1 *and* f_2 *are central in* $Hom(H \otimes H, A)$

(ii) ω *is a normalized invertible cocycle of the ideal* $\langle f_1 * f_2 \rangle \subset Hom(H \otimes H, A)$, *i.e.,* ω *satisfies conditions (1) and (2) and has a convolution inverse* ω' *in* $\langle f_1 * f_2 \rangle$.

(iii) $(h \cdot (k \cdot 1_A)) = \sum (h_{(1)} \cdot 1_A)(h_{(2)} k \cdot 1_A)$, *for every* $h, k \in H$.

Partially Cleft Extensions

Definition

Let B be a right H-comodule algebra with coaction given by $\rho : B \to B \otimes H$ and let $A = B^{coH}$. The H-extension $A \subset B$ is partially cleft if there is a pair of k-linear maps $\gamma, \gamma' : H \to B$ such that

(i) $\gamma(1) = 1_B$

(ii) $\rho \circ \gamma = (\gamma \otimes I) \circ \Delta$, *and* $\rho \circ \gamma' = (\gamma' \otimes S) \circ \Delta^{op}$.

(iii) $\gamma' * \gamma$ *commute with every element of* A, $\gamma * \gamma'$ *is a central idempotent in the convolution algebra* $Hom(H, A)$.

(iv) $\sum b^{(0)} \gamma'(b^{(1)}) \gamma(b^{(2)}) = b$, $\forall b \in B$

(v) $\sum \gamma(h_{(1)} k_{(1)}) \gamma'(k_{(2)}) \gamma'(h_{(2)}) \gamma(h_{(3)}) \gamma(k_{(3)}) = \sum \gamma(h_{(1)}) \gamma'(h_{(2)}) \gamma(h_{(3)} k)$, $\forall h, k \in H$.

Partially Cleft Extensions, II

Theorem

(1) *The partial crossed product $A\#_\omega H$ is an H-partially cleft extension of A.*

(2) *If $A \subset B$ is an H-partially cleft extension, then there is a twisted partial action of H on A such that $B \cong A\#_\omega H$.*

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Globalization

Definition

Let $(A, (w, w'))$, $(A', (v, v'))$ be two symmetric twisted partial actions of H. A map $\varphi : A \to A'$ is an equivalence of twisted partial actions if

(i) φ *is an algebra monomorphism;*

(ii) $\varphi(h \cdot a) = h \cdot \varphi(a)$;

(iii) $\varphi(w) = v$, $\varphi(w') = v'$.

Globalization, II

Definition

Consider a symmetric twisted partial action of H on A with the pair (w, w'). A globalization of A is a pair (B, φ), where B is a twisted H-module algebra, possibly non-unital, with invertible cocycle u, and $\varphi : A \to B$ is an algebra monomorphism such that

(i) *$\varphi(A)$ is a right ideal in B;*

(ii) *B is the subalgebra generated by $H \triangleright \varphi(A)$*

(iii) *$\varphi : A \to \varphi(A)$ is an equivalence of twisted partial actions of H, where the partial twisted action on $\varphi(A)$ is induced by the global twisted action on$B$.*

Globalization, III

Theorem

Let H be a Hopf algebra and a symmetric twisted partial action on A given by the pair (w, w'). A has a globalization if and only if there is a convolution invertible map $\tilde{\omega} : H \otimes H \to A$ satisfying

$$\sum (h_{(1)} \cdot \tilde{\omega}(k_{(2)}, l_{(1)}))\tilde{\omega}(h_{(2)}, k_{(2)} l_{(2)}) =$$
$$= \sum (h_{(1)} \cdot \mathbf{1}_A)\tilde{\omega}(h_{(2)}, k_{(1)})\tilde{\omega}(h_{(3)} k_{(2)}, l_{(2)})$$

with $\tilde{\omega}(1, h) = \tilde{\omega}(h, 1) = \epsilon(h)\mathbf{1}_A$, such that

$$\begin{aligned}
\omega(h, k) &= \sum (h_{(1)} \cdot (k_{(1)} \cdot \mathbf{1}_A))\tilde{\omega}(h_{(2)}, k_{(2)}) \\
\omega'(h, k) &= \sum \tilde{\omega}^{-1}(h_{(2)}, k_{(2)})(h_{(2)} \cdot (k_{(2)} \cdot \mathbf{1}_A))
\end{aligned}$$

Perspectives

- To find a better set of axioms to describe partially cleft extensions..
- To find out whether is it always possible to globalize, or there is an obstruction.
- A cohomological setting for these partial cocycles.
- To put partial actions into a broader abstract context (Hopf algebroids).
- To extend to the non unital case, to go to Multiplier Hopf Algebras.

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references

- M.M.S. Alves, E. Batista, "Enveloping Actions for Partial Hopf Actions", Comm. Algebra, v. 38, (2010) 2872-2902.
- M.M.S. Alves, E. Batista, M. Dokuchaev, A. Paques: "Twisted partial actions of Hopf algebras", (to appear).
- S. Caenepeel, K. Janssen, "Partial (co)actions of Hopf algebras and partial Hopf-Galois theory", Comm. Algebra 36 (2008), 2923-2946.
- M. Dokuchaev, R. Exel: "Associativity of Crossed Products by Partial Actions, Enveloping Actions and Partial Representations" Trans. Amer. Math. Soc. 357 (5) (2005) 1931-1952.
- M. Dokuchaev, R. Exel and J.J. Simón, "Crossed products by twisted partial actions and graded algebras", J. Algebra 320 (2008), 3278-3310.
- M. Dokuchaev, R. Exel and J.J. Simón, "Globalization of twisted partial actions", Trans. Amer. Math. Soc. 362 (2010) 4137-4160.

Thank you very much!