Hopf semialgebras

Jawad Y. Abuhlail (King Fahd University of Petroleum & Minerals, Saudi Arabia)
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In this talk, we introduce and investigate the notions of semibialgebras and Hopf semialgebras over semirings. We also investigate several related categories of Doi-Koppinen semimodules.

Hopf Semialgebras

Hopf algebras and tensor categories
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King Fahd University of Petroleum & Minerals

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Outline

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In this talk, we introduce and investigate the notions of semibialgebras and Hopf semialgebras over semirings. We also prove the Fundamental Theorem of Hopf Semialgebras.

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- $(L, \vee, \wedge)$ *is a b.d. lattice having unique minimal & unique maximal* $\Leftrightarrow$ *L is a comm. idempotent simple semiring.*

Examples & Remarks

- The semiring $\mathbb{R}_{\max} := (\mathbb{R} \cup \{-\infty\}, max, +)$, which is a idempotent dequantization of $\mathbb{R}^+$, plays an important role in Tropical Geometry and Idempotent Analysis (e.g. Mikhalkin 2006; Litvinov & Maslov 2005, Litvinov 2010).

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- Semirings $\leftrightarrow$ additive algebraic monads on **Set** (Durov 2007).

Semimodules

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- A comprehensive literature review on semirings and their applications is provided by K. Głazek (2002).

Factorization Structures

Definition

(Adámek, Herrlich and Strecker 2004) Let $\mathbf{E} \subseteq \mathbf{Epi}$ *and* $\mathbf{M} \subseteq \mathbf{Mono}$. *We call* $(\mathbf{E}, \mathbf{M})$ *a factorization structure for morphisms in* $\mathfrak{C}$ *provided that*

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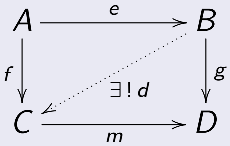
3. $\mathfrak{C}$ *has the unique* $(\mathbf{E}, \mathbf{M})$*-diagonalization property*



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Example

An epimorphism which is not surjective:

$$h : \mathbb{N}_0 \times \mathbb{N}_0 \longrightarrow \mathbb{N}_0 \times \mathbb{N}_0, (n, m) \mapsto (2n + m, n).$$

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- not homological (in the sense of Borceux & Bourn: 2004; a category which is pointed, regular & protomodular)

Subtractive Subsemimodules

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3. *$\xi : M \to M^\Delta$ is injective, where $M^\Delta := (M \times M)/W$.*

Cancellative Semimodules

Definition

Let M be an S-semimodule and consider the S-congruence relation

$$m[\equiv]_{\{0\}}m' \Leftrightarrow m + m'' = m' + m''; \;\; m'' \in M.$$

Then we have a cancellative S-semimodule

$$\mathfrak{c}(M) := M/[\equiv]_{\{0\}} = \{[m]_{\{0\}} : m \in M\}. \tag{1}$$

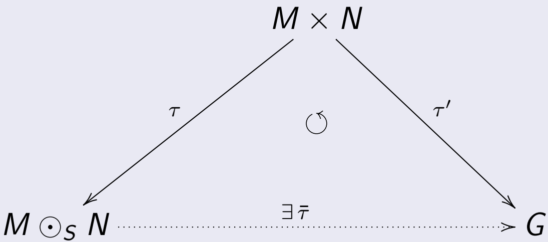
Moreover, we have a canonical surjection $\mathfrak{c}_M : M \longrightarrow \mathfrak{c}(M)$ with

$$\delta(M) := \mathrm{Ker}(\mathfrak{c}_M) = \{m \in M \mid m + m' = m'; \;\; m' \in M\}.$$

Takahashi's Tensor Product 1982

Universal Property

Let M be a right S-semimodule and N a left S-semimodule. For every cancellative commutative monoid G:



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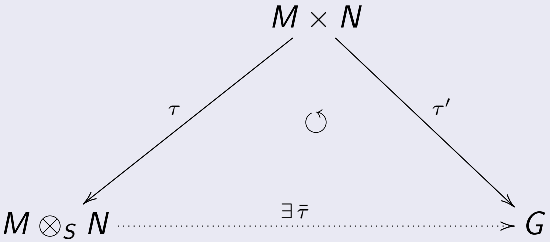

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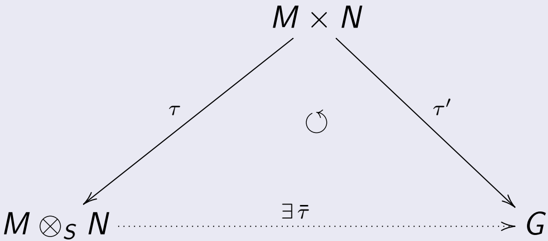

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Exactness in Non-exact Categories ... (under preparation)

Definition

Let $\mathfrak{C}$ be any pointed category. We call a sequence $A \xrightarrow{f} B \xrightarrow{g} C$ exact iff

- $f = \ker(g) \circ f'$, *where* $(f', \ker(g)) \in \mathbf{E} \times \mathbf{M}$;
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$$f(m) = f(m') \Rightarrow m+k = m'+k' \text{ for some } k, k' \in \mathrm{Ker}(f).$$

Exact Sequences

We call a sequence of S-semimodules $L \xrightarrow{f} M \xrightarrow{g} N$:
exact iff $f(L) = \mathrm{Ker}(g)$ and g is steady;
proper-exact iff $f(L) = \mathrm{Ker}(g)$ (exact in Patchkoria 2003);
semi-exact iff $\widehat{f(L)} = \mathrm{Ker}(g)$ (exact in Takahashi 1981);
weakly-exact iff $\widehat{f(L)} = \mathrm{Ker}(g)$ and g is steady (PD2006).

Semialgebras

Assumption

From now on, S is a commutative semiring, so that $({}_S\mathbb{S}, \otimes, S)$ is a symmetric monoidal category.

Definition

With an S-semialgebra, we mean a triple (A, μ_A, η_A), where A is a S-semimodule and

$$\mu : A \otimes_S A \longrightarrow A \text{ and } \eta : S \longrightarrow A$$

are S-linear morphisms such that

$$\mu_A \circ (\mu_A \otimes_S \mathrm{id}_A) = \mu_A \circ (\mathrm{id}_A \otimes_S \mu_A);$$
$$\mu_A \circ (\eta_A \otimes_S \mathrm{id}_A) = \vartheta^l_A \ \& \ \mu_A \circ (\mathrm{id}_A \otimes_S \eta_A) = \vartheta^r_A.$$

Semicoalgebras ... (Brussels, 2008)

Definition

An *S-semicoalgebra* *is an S-semimodule C associated with*

$$\Delta_C : C \to C \odot_S C \text{ and } \varepsilon_C : C \to S,$$

$$\begin{array}{ccc}
C & \xrightarrow{\Delta_C} & C \odot_S C \\
{\scriptstyle \Delta_C}\downarrow & & \downarrow{\scriptstyle \mathrm{id}_C \odot_S \Delta_C} \\
C \odot_S C & \xrightarrow[\Delta_C \odot_S \mathrm{id}_C]{} & C \odot_S C \odot_S C
\end{array}$$

$$\begin{array}{ccccc}
C \odot_S C & \xleftarrow{\Delta_C} & C & \xrightarrow{\Delta_C} & C \odot_S C \\
{\scriptstyle \varepsilon_C \odot_S \mathrm{id}_C}\downarrow & & \downarrow{\scriptstyle \mathfrak{c}_C} & & \downarrow{\scriptstyle \mathrm{id}_C \odot_S \varepsilon_C} \\
S \odot_S C & \xrightarrow[\vartheta_C^l]{} & \mathfrak{c}(C) & \xleftarrow[\vartheta_C^r]{} & C \odot_S S
\end{array}$$

Semicoalgebras ... (Almeria, 2011)

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An *S-semicoalgebra* *is an S-semimodule C associated with*

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$$\begin{CD} C @>\Delta_C>> C \otimes_S C \\ @V\Delta_C VV @VV\mathrm{id}_C \otimes_S \Delta_C V \\ C \otimes_S C @>>\Delta_C \otimes_S \mathrm{id}_C> C \otimes_S C \otimes_S C \end{CD}$$

$$\begin{array}{ccccc} C \otimes_S C & \xleftarrow{\Delta_C} & C & \xrightarrow{\Delta_C} & C \otimes_S C \\ \downarrow{\scriptstyle \varepsilon_C \otimes_S \mathrm{id}_C} & \nearrow{\scriptstyle \vartheta^l_C} & & \nwarrow{\scriptstyle \vartheta^r_C} & \downarrow{\scriptstyle \mathrm{id}_C \otimes_S \varepsilon_C} \\ S \otimes_S C & & & & C \otimes_S S \end{array}$$

Examples

Example

For any set X, the free S-semimodule $S^{(X)}$ is an S-semicoalgebra:

$$\Delta(x) = x \otimes_S x \text{ and } \varepsilon(x) = 1_S \text{ for all } x \in X.$$

Semibialgebras

Definition

With an S-semibialgebra, we mean a datum $(H, \mu, \eta, \Delta, \varepsilon)$, *where* (H, μ, η) *is an* S*-semialgebra and* (H, Δ, ε) *is an* S*-semicoalgebra such that*

$$\Delta : H \longrightarrow H \otimes_S H \text{ and } \varepsilon : H \longrightarrow S$$

are morphisms of S*-semialgebras, or equivalently*

$$\mu : H \otimes_S H \longrightarrow H \text{ and } \eta : S \longrightarrow H$$

are morphisms of S*-semicoalgebras.*

Doi-Koppinen semimodules

Let B be an S-semibialgebra. As in the classical case, one can consider several examples of Doi-Koppinen semimodules.

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- (M, ρ_M) is a right B-semimodules;
- (M, ϱ^M) is a right B-semicomodule;
- $\varrho^M(mb) = \sum m < 0 > b_1 \otimes_S m < 1 > b_2 \, \forall \, m \in M$ and $b \in B$.

Automata ... Worthington 2009

Definition

A left automaton $(M, A, \triangleleft, s, \gamma)$ *consists of:*

1. *an* S*-semialgebra* A *and a left* A*-semimodule* $(M, \triangleleft)$*;*
2. *an element* $s \in M$*, called start vector;*
3. *an* S*-linear map* $\gamma : M \longrightarrow S$*.*

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A morphism of left automata

$$(\varphi, \lambda) : (M, A, \triangleleft, s, \gamma) \to (M', A', \triangleleft', s', \gamma'), m \mapsto \varphi(m)$$

is a pair consisting of a morphism of S*-semialgebras* $\lambda : A \to A'$ *and an* A*-linear map* $\varphi : M \to M'$*, s.t.:*

$$\varphi(s) = s' \text{ and } \gamma(m) = \gamma(\varphi(m)) \text{ for all } m \in M.$$

Application ... Worthington 2009

Definition

The language accepted by the left automaton $(M, A, \lhd, s, \gamma)$ *is the* S*-linear map*

$$\rho_M : A \longrightarrow S, a \mapsto \gamma(a \lhd s) \text{ for all } a \in A.$$

Application ... Worthington 2009

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Example

Let B *be an* S*-semibialgebra and* $(M, B, \lhd, s, \gamma)$, $(M', B, \lhd', s', \gamma')$ *be left automata. Then* $(M \otimes_S^b M', B, \lhd_{M \otimes_S^b M'}, s \otimes_S s', \gamma \otimes_S \gamma')$ *is a left automata.*

Application ... Worthington 2009

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$$b \lhd_{M \otimes_S M'} m \otimes_S m' := \sum b_1 m \otimes_S b_2 m'.$$

Application ... Worthington 2009

Definition

The language accepted by the left automaton $(M, A, \triangleleft, s, \gamma)$ *is the S-linear map*

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Example

Let B be an S-semibialgebra and $(M, B, \triangleleft, s, \gamma)$, $(M', B, \triangleleft', s', \gamma')$ *be left automata. Then* $(M \otimes_S^b M', B, \triangleleft_{M \otimes_S^b M'}, s \otimes_S s', \gamma \otimes_S \gamma')$ *is a left automata.*

$$b \triangleleft_{M \otimes_S M'} m \otimes_S m' := \sum b_1 m \otimes_S b_2 m'.$$

Moreover, $\rho_{M \otimes_S^b M'} = \rho_M * \rho_{M'}$.

Classical Examples

Example

Let $(G,\cdot,e)$ be a monoid. Then $S[G]$ is an S-semibialgebra with:

$$\begin{array}{ccccccccc} \Delta & : & S[G] & \longrightarrow & S[G]\otimes_S S[G], & g & \longmapsto & g\otimes_S g; \\ \varepsilon & : & S[G] & \longrightarrow & S, & g & \longmapsto & 1_S. \end{array}$$

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Example

$(S[x], \Delta_1, \varepsilon_1)$ is an S-semibialgebra with:

$$\Delta_1(x^n) := x^n \otimes_S x^n \text{ and } \varepsilon_1(x^n) = 1_S.$$

Hopf Semialgebras

Definition

A Hopf S-semialgebra is an S-semibialgebra H, along with an S-linear morphism $S : H \to H$ (called the antipode of H), such that

$$\sum S(h_1)h_2 = \varepsilon(h)1_H = \sum h_1 S(h_2) \text{ for all } h \in H.$$

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Example

Consider the commutative semialgebra $\mathbb{B} := \{0, 1\}$ *with* $1 + 1 = 1$. *Then* $\mathbb{B}$ *is a Hopf semialgebra:*

$$\begin{array}{rclrcl} \Delta(0) & = & 0 \otimes 0, & \Delta(1) & = & 1 \otimes 1 \\ \varepsilon(0) & = & 0, & \varepsilon(1) & = & 1 \\ S(0) & = & 0, & S(1) & = & 1. \end{array}$$

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Let $(G, \cdot, e)$ be a group. Then $S[G]$ is a Hopf S-semialgebra:

$$\begin{array}{lllllll} \Delta & : & S[G] & \longrightarrow & S[G] \otimes_S S[G], & g & \mapsto & g \otimes_S g; \\ \varepsilon & : & S[G] & \longrightarrow & S, & g & \mapsto & 1_S. \\ S & : & S[G] & \longrightarrow & S[G], & g & \mapsto & g^{-1}. \end{array}$$

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Example

$(S[x],\mu,\eta,\Delta_2,\varepsilon_2,S)$ *is a Hopf S-semialgebra:*

$$\Delta_2(x^n) := \sum_{k=0}^{n}\binom{n}{k} x^k \otimes_S x^{n-k},\ \varepsilon_2(x^n) = \delta_{n,0}$$

$$S : H \rightarrow H,\ \ S(x^n) = (-1)^n x^n.$$

Classical Examples ... continued

Example

$(S[x, x^{-1}], \mu, \eta, \Delta, \varepsilon, S)$ *is a Hopf S-semialgebra:*

$$\begin{aligned} \Delta(x^z) &= x^z \otimes_S x^z; \\ \varepsilon(x^z) &= 1_S; \\ S(x^z) &= x^{-z}. \end{aligned}$$

Fundamental Theorem of Hopf Semialgebras

Theorem

Consider a S-semibialgebra B and the corresponding category of Hopf modules $\mathbb{S}_B^B$. The following are equivalent:

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3. $B \otimes_S B \simeq B \otimes_S^c B$ *in* $\mathbb{S}_B^B$*;*
4. $Hom_B^B(B, -) : \mathbb{S}_B^B \to \mathbb{S}_S$ *is an equivalence (with inverse* $- \otimes_S B$*).*

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