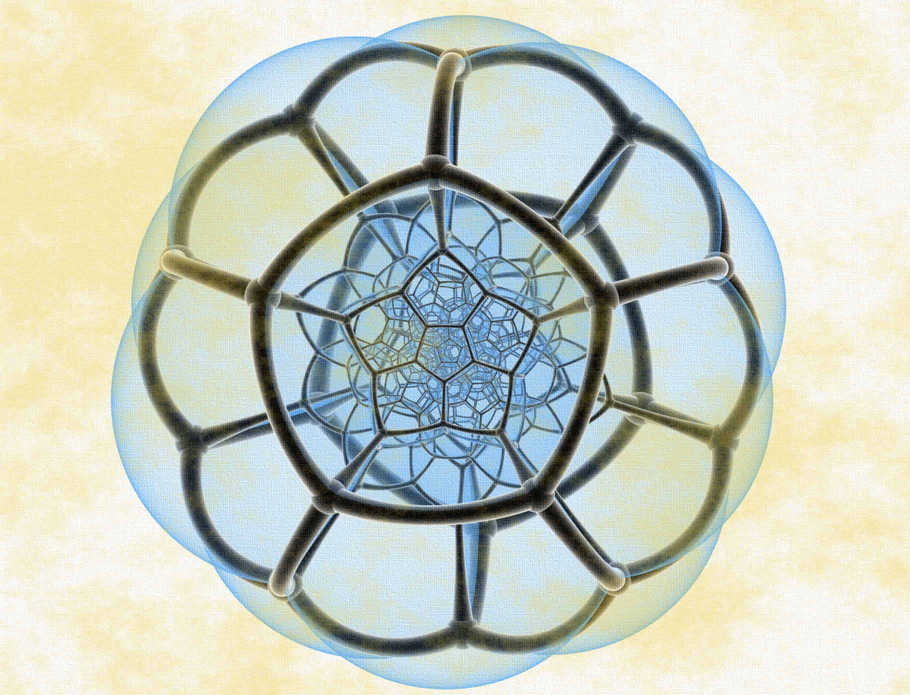
Exploiting symmetry on the Universal Polytope

Julian Pfeifle

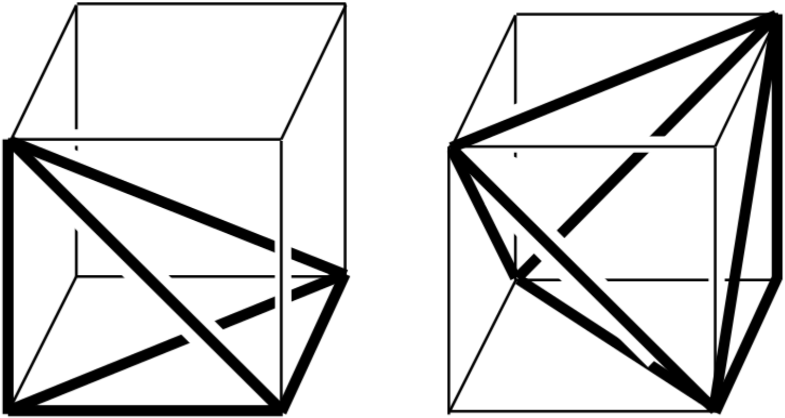
(Universitat Politècnica de Catalunya)

VIII JMDA Almería



Motivation

What is the minimal number of simplices needed
to triangulate a convex polytope?



Motivation

Officially,

this is important in

- algorithms for iteratively finding fixed points (i.e., Nash equilibria)
- financial applications

For me,

this is important because

- every time people have tried to solve the problem, interesting mathematics came out

Overview

Previous work:

1. People have looked at cubes and products of simplices
2. Focus on explicit dimensions: Lower bounds for $2 \leq d \leq 11$
3. Focus on asymptotics: product constructions, hyperbolic geometry
4. some structural insights (e.g., the Universal Polytope)

Today:

1. Reduction of symmetry
2. Application to triangulations of manifolds

Explicit lower bounds

[Hughes 1993-4], [Hughes & Anderson 1996] $\square^d$

Dimension d of cube	3	4	5	6	7	8	9 ...
min # simplices $\sigma(\square^d)$	5	16	67	308	1493	≥ 5522	$\geq 26\,593$ ...

[Seacrest & Su, 2009] $\Delta^s \times \Delta^t$

Explicit lower bounds on $\sigma(\Delta^s \times \Delta^t)$ for $s + 2t \leq 12$

[Smith, 2000] $\square^{d \gg 0}$

$$\sigma(\square^d) \geq \frac{\text{Hvol(regular ideal } \square^d)}{\text{Hvol(regular ideal } \Delta^d)} \geq \frac{1}{2} 6^{n/2} (n+1)^{-\frac{n+1}{2}} n!$$

Explicit upper bounds

[Haiman, 1991] $\square^{d \gg 0}$

$$\sigma(\square^{d \gg 0}) \ \leq \ \rho^d \cdot d!, \qquad \text{for some } \rho < 1.$$

uses a product formula, and induction.

Explicit upper bounds

[Haiman, 1991] $\square^{d \gg 0}$

$$\sigma(\square^{d \gg 0}) \ \leq \ \rho^d \cdot d!, \qquad \text{for some } \rho < 1.$$

uses a product formula, and induction.

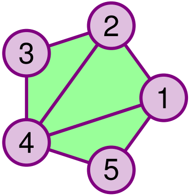
[Orden & Santos, 2003] $\square^{d \gg 0}$

$$\rho \leq 0.8159$$

- induction start: $\square^3 \times \triangle^2$
- use CPLEX to solve a linear program with 74 400 variables
- 37 CPU hours on a SUN UltraSparc

The Universal Polytope

$$\chi(\mathcal{T}) = \begin{array}{c} \\ \left[\right. \end{array}\begin{array}{cccccccccc} 123 & 124 & 125 & 134 & 135 & 145 & 234 & 235 & 245 & 345 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 \end{array}\begin{array}{c} \\ \left. \right] \end{array} \in \{0,1\}^{\binom{5}{3}}$$



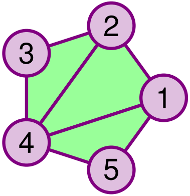
$$\mathcal{U}(\mathcal{A}) \quad = \quad \text{conv}\left\{\chi_{\mathcal{T}} : \mathcal{T} \text{ triang of } \mathcal{A}\right\} \subset \mathbb{R}^{\binom{n}{d+1}}$$

We need to understand $\mathcal{U}(\mathcal{A})$.

For example, what are its equations?

The Universal Polytope

	123	124	125	134	135	145	234	235	245	345	
$\chi(\mathcal{T}) =$ [	0	1	0	0	0	1	1	0	0	0	] $\in \{0,1\}^{\binom{5}{3}}$



$$\mathcal{U}(\mathcal{A}) = \text{conv}\,\{\chi_\mathcal{T} : \mathcal{T} \text{ triang of } \mathcal{A}\} \subset \mathbb{R}^{\binom{n}{d+1}}$$

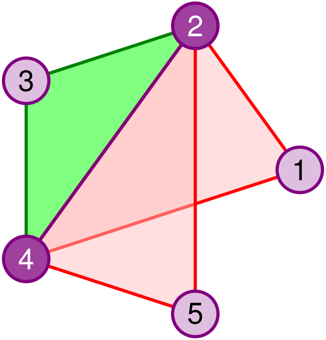
$$\sigma(\mathcal{A}) = \text{min. cardinality of a triangulation}$$

$$= \min_{\mathcal{T}} \left\{ \sum_{\Delta \in \mathcal{T}} x_\Delta : x \in \mathcal{U}(\mathcal{A}) \right\}$$

We need to understand $\mathcal{U}(\mathcal{A})$.

For example, what are its equations?

The Cocircuit Equations



$$L \in \{0, \pm 1\}^{\Sigma_{\text{int}}^{d-1} \times \Sigma^d}$$

$$L\,\chi_{\mathcal{T}}^{\top} \;=\; 0$$

These generate all the linear relations among the entries of $\chi_{\mathcal{T}}$.

$e = \mathbf{24}: \quad x_{234} - x_{124} - x_{245} \;=\; 0$

Symmetry

Optimization over $\mathcal{U}(\square^d)$ is only feasible for $d \leq 5$.

- Hughes & Anderson consider equivalence classes of simplices in such a way that non-congruent simplices become equivalent
- However, little structural insight, and no asymptotics

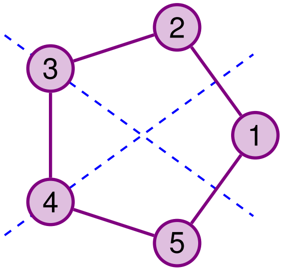
Our approach: form equivalence classes of simplices w.r.t. $\mathrm{Aut}(\square^d)$

$$\Delta_1 \cong \Delta_2 \quad \text{iff} \quad \exists g \in \mathrm{Aut}(\square^d) : g(\Delta_1) = \Delta_2$$

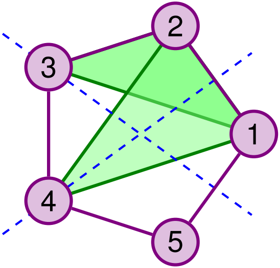
Exploiting symmetry reduces the dimension

What are the images of the cocircuit equations?

Exploiting symmetry: $G = \langle (12)(35), (15)(24) \rangle$



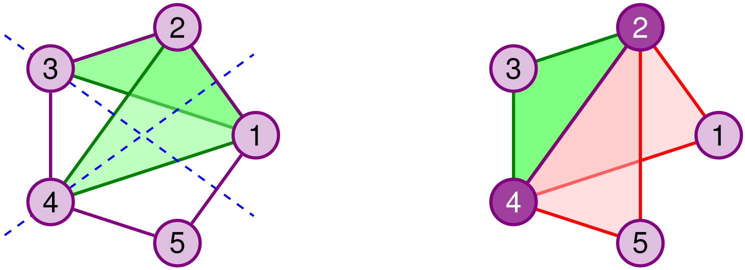
Exploiting symmetry: $G = \langle (12)(35), (15)(24) \rangle$



symmetry classes of triangles:

$$\overline{123} = \{123, 125, 145, 234, 345\},$$
$$\overline{124} = \{124, 245, 134, 135, 235\}$$

Exploiting symmetry: $G = \langle (12)(35), (15)(24) \rangle$

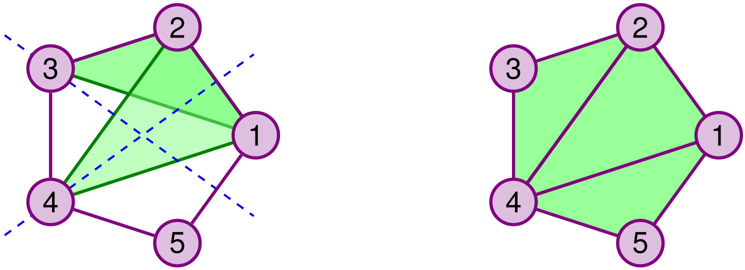


symmetry classes of triangles:

$$\overline{123} = \{123, 125, 145, 234, 345\},$$
$$\overline{124} = \{124, 245, 134, 135, 235\}$$

cocircuit relations: $x_{234} = x_{124} + x_{245} \quad \Longrightarrow \quad y_{\overline{123}} = 2y_{\overline{124}}$

Exploiting symmetry: $G = \langle (12)(35), (15)(24) \rangle$



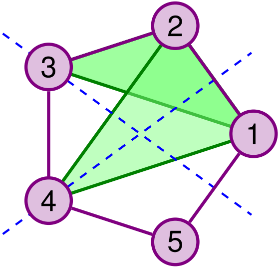
symmetry classes of triangles:

$$\overline{123} = \{123, 125, 145, 234, 345\},$$
$$\overline{124} = \{124, 245, 134, 135, 235\}$$

cocircuit relations: $x_{234} = x_{124} + x_{245} \quad \Longrightarrow \quad y_{\overline{123}} = 2y_{\overline{124}}$

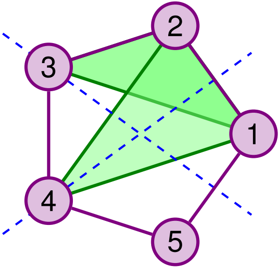
volume relation: $x_{234} + x_{124} + x_{145} = 1 \quad \Longrightarrow \quad y_{\overline{123}} + y_{\overline{124}} = 3$

Exploiting symmetry: Setting up a linear program



$$
\begin{aligned}
\min \quad & y_{\overline{123}} + y_{\overline{124}} \\
s.t. \quad & y_{\overline{123}} = 2y_{\overline{124}} && \text{cocircuit equation} \\
& y_{\overline{123}} + y_{\overline{124}} = 3 && \text{volume equation} \\
& y_{\overline{123}},\ y_{\overline{124}} \geq 0
\end{aligned}
$$

Exploiting symmetry: Setting up a linear program



$$
\begin{aligned}
\min\ & y_{\overline{123}} + y_{\overline{124}} \\
s.t.\ & y_{\overline{123}} = 2y_{\overline{124}} && \text{cocircuit equation} \\
& y_{\overline{123}} + y_{\overline{124}} = 3 && \text{volume equation} \\
& y_{\overline{123}},\ y_{\overline{124}} \geq 0 && y_{\overline{123}} = 2,\ y_{\overline{124}} = 1
\end{aligned}
$$

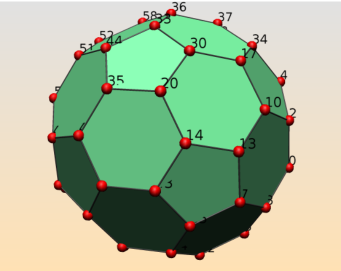
Implementation

- Calculation with symmetry groups:
  `polymake`, permlib

- Symmetry groups of regular polytopes:

Need to calculate exactly with
quadratic extensions $\mathbb{Q}[\sqrt{d}]$

Implemented this in the
upcoming **`polymake 2.13`**



- Payoff: Can calculate lower bounds for
  the simplexity of quotient manifolds
  (e.g., Poincaré homology 3-sphere = dodecahedron mod
  identifications on the boundary)

Results

- Remember Orden & Santos:
  $\square^3 \times \triangle^2$ needs 74 400 variables, 37 CPU hours

Results

- Remember Orden & Santos:
  $\square^3 \times \triangle^2$ needs 74 400 variables, 37 CPU hours

  Our approach: 323 variables and 414 constraints; just under 4 minutes (exactly), or 0.10 seconds (floating-point)

Results

- Remember Orden & Santos:
  $\square^3 \times \triangle^2$ needs 74 400 variables, 37 CPU hours

  Our approach: 323 variables and 414 constraints; just under 4 minutes (exactly), or 0.10 seconds (floating-point)

- Simplexity of $\square^7$: under 1 hour

Results

- Remember Orden & Santos:
  $\square^3 \times \triangle^2$ needs 74 400 variables, 37 CPU hours

  Our approach: 323 variables and 414 constraints; just under 4 minutes (exactly), or 0.10 seconds (floating-point)

- Simplexity of $\square^7$: under 1 hour

- Simplexity of $\square^8$: **???**
  (have enumerated all 41 258 870 representatives of the 4×10^{14} simplices of dim 7; occupy 1GB)

Results

- Remember Orden & Santos:
  $\square^3 \times \triangle^2$ needs 74 400 variables, 37 CPU hours

  Our approach: 323 variables and 414 constraints; just under 4 minutes (exactly), or 0.10 seconds (floating-point)
- Simplexity of $\square^7$: under 1 hour
- Simplexity of $\square^8$: **???**
  (have enumerated all 41 258 870 representatives of the 4×10^{14} simplices of dim 7; occupy 1GB)
- Simplexity of Davis' 4-manifold (120-cell mod identifications): **???**
  (have enumerated all 44 238 243 representatives of 4-simplices; occupy 773M)

Implementation

```
time polymake 'my $c=product(cube(3),simplex(2));
               linear_symmetries($c,1);
               print $c->SIMPLEXITY_LOWER_BOUND;'
```

polymake

```
polymake: used package cddlib
  Implementation of the double description method of Motzkin et al.
  Copyright by Komei Fukuda.
  http://www.ifor.math.ethz.ch/~fukuda/cdd_home/cdd.html
38
real 3m26.755s
user 3m26.617s
sys 0m0.084s
```

```
polymake 'truncated_icosahedron()->VISUAL;'
```

```
polymake 'print truncated_icosahedron()->VOLUME;'
125/4 + 43/4 r5
```

Outlook

- Complete calculations for Davis' manifold
- Triangulations with other special properties:
  bipartite dual graph (interesting for lower bounds for the number of real roots of certain sparse polynomial systems)
- Different direction: Sharpen asymptotic lower bounds using hyperbolic geometry

Gracias!