# Watching Systems in Complete Bipartite Graphs

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# Outline

# Detection devices

- Detection devices located at some vertices of a graph
- Detect and locate an object placed at any vertex of a graph
- Dominating/total dominating sets
- Locating sets

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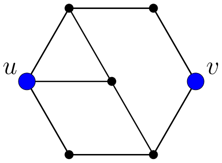
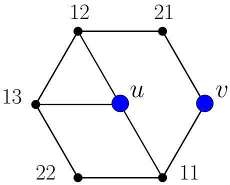
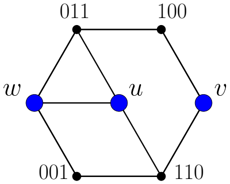
# Detection devices and graphs



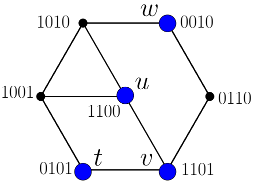

Dominating set




Locating set




Locating-dominating set




Identifying set

# Definitions

$G = (V, E)$ graph,

- $N(u) = \{v : uv \in E\}$
- $N[u] = \{u\} \cup N(u)$
- *twin vertices*: $N[u] = N[v]$
- *twin-free* graph: it has no pair of twin vertices
- *dominating set*: $S \subseteq V$ s.t. for all $v \in V \setminus S$, $S \cap N(v) \neq \emptyset$
- *dominating number*, $\gamma(G)$: minimum size of a dominating set of $G$

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# Identifying codes

## [Karpovsky, Chakrabarty, Levitin, 1998]

*Identifying code* in a graph $G = (V, E)$:

$S \subseteq V$ s.t. the sets $N[v] \cap C$, $v \in V(G)$, are all nonempty and distinct.

- *label* of vertex $v$: $L_C(v) = N[v] \cap C$
- *identifying number*, $i(G)$: minimum size of an identifying code of $G$
- Identifying codes exist only in twin-free graphs.

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# Watching systems

[Auger, Charon, Hudry, Lobstein, 2010]

*Watching system* in a graph $G = (V, E)$ graph:

$W = \{w_1, w_2, \ldots, w_k\}$ where $w_i = (I(w_i), A(w_i))$, with $I(w_i) = v_i \in V(G)$ and $A(w_i) \subseteq N[v_i]$, for all $i \in \{1, 2, \ldots, k\}$, s.t. the sets $L_W(v) = \{w \in W : v \in A(w_i)\}$ are all nonempty and distinct.

- $w_i$ is a *watcher* located at vertex $I(w_i)$ that checks its *watching zone*, $A(w_i)$
- $L_W(v)$ is the label of vertex $v$

Several watchers at the same vertex, each watcher checks its watching zone

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- *watching number*, $w(G)$: minimum size of a watching system of $G$
- *minimum watching system*: watching system of cardinality $w(G)$
- Watching systems exist for all graphs
- $w(G) \leq i(G)$ if there exists at least an identifying code in $G$
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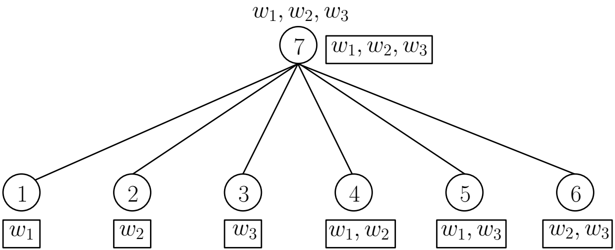
# Example

$G = K_{1,6}$: $i(G) = 6$, $w(G) = 3$

$W = \{w_1, w_2, w_3\}$, $l(w_i) = 7$

$A(w_1) = \{1,4,5,7\}$, $A(w_2) = \{2,4,6,7\}$, $A(w_3) = \{3,5,6,7\}$




$L_W(1) = \{w_1\}$, $L_W(2) = \{w_2\}$, $L_W(3) = \{w_3\}$, $L_W(4) = \{w_1, w_2\}$,
$L_W(5) = \{w_1, w_3\}$, $L_W(6) = \{w_2, w_3\}$, $L_W(7) = \{w_1, w_2, w_3\}$.

# General bounds of the watching number

- $w(G) \geq \lceil \log_2(n+1) \rceil$
- Complete graphs, stars, graphs s.t. $\Delta = n - 1$ attain this bound

- $w(G) \geq \gamma(G)$
- $w(G) \leq \gamma(G) \lceil \log_2(\Delta + 2) \rceil$

- $w(G) \leq i(G)$ , if $G$ is twin-free
- $w(G) \leq w(H)$ for any spanning subgraph $H$ of $G$

- $w(G) \leq \frac{2n}{3}$, if $G$ is a connected graph of order 3 or $\geq 5$ [Auger, Charon, Hudry, Lobstein, to appear]

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# Watching number and identifying number of some families

$$w(P_n) = \left\lceil \frac{n+1}{2} \right\rceil \qquad i(P_n) = \left\lceil \frac{n+1}{2} \right\rceil$$

$$w(C_n) = \begin{cases} 3 & \text{, if } n = 4; \\ \lceil \frac{n}{2} \rceil & \text{, otherwise.} \end{cases} \qquad i(C_n) = \begin{cases} 3, & \text{if } n = 4, 5; \\ \frac{n}{2}, & \text{if } n \geq 6 \text{ even;} \\ \frac{n+3}{2}, & \text{if } n \geq 7 \text{ odd.} \end{cases}$$

# Complete bipartite graphs

$K_{r,s}$, $2 \leq r \leq s$

- $\gamma(K_{r,s}) = 2$
- $i(K_{r,s}) = r + s - 2$

$W = \{w_i \,:\, i \in [m]\}$ watching system in $K_{r,s}$

- $V(K_{r,s}) = V_1 \cup V_2$, $|V_1| = r$, $|V_2| = s$
- $\mathcal{L}(W) = \{l(w_i) \,:\, i \in [m]\} \subseteq V$
- $\mathcal{L}_1(W) = \mathcal{L}(W) \cap V_1$, $\mathcal{L}_2(W) = \mathcal{L}(W) \cap V_2$,

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# Bounds

$$w_0(r,s) = \lceil \log_2(r+s+1) \rceil$$

Bounds:

- $w_0(r,s) \leq w(K_{r,s}) \leq \lceil \log_2 r \rceil + \lceil \log_2 s \rceil$

Both bounds are tight:

- $w(K_{3,16}) = w_0(3,16) = 5$
- $w(K_{8,11}) = \lceil \log_2 8 \rceil + \lceil \log_2 11 \rceil = 7$

Particular case:

- $w(K_{2,s}) = w_0(2,s) = \lceil \log_2(s+3) \rceil$

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# Watching Systems in Complete Bipartite Graphs

Consider $K_{r,s}$, $2 \leq r \leq s$:

- If a watching system has 2 watchers at a same vertex, we obtain another watching system by placing one of them at another vertex of the same stable set
- A watching system with all watchers located in the same stable set has size at least $\max\{r, \lceil \log_2(r+s+1) \rceil\}$
- A watching system with at least a watcher in each stable set has size $> w_0(r,s)$

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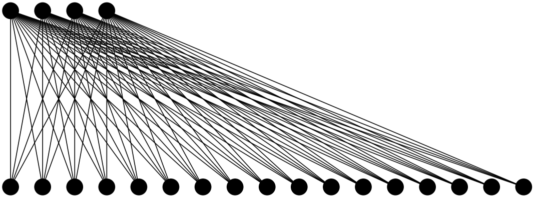
# Attaining the lower bound



If $2 \leq r \leq s$,

- If $K_{r,s} \neq K_{5,5}$, $w(K_{r,s}) = w_0(r,s)$ if and only if $r \leq w_0(r,s)$.

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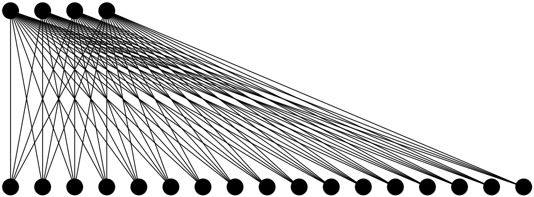

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# Not attaining the lower bound

If $r > w_0(r,s)$,

- There is a minimum watching system $W$ satisfying $|\mathcal{L}_1(W)| \geq |\mathcal{L}_2(W)|$
- $w(K_{r,s}) = \min\{m : m = h+k, r \leq k+2^h-1, s \leq h+2^k-1\}$
- If $6 \leq r = s$, then $w(K_{r,r}) \neq w_0(r,r)$
- For each $r \geq 3$, there is a minimum watching system of $K_{r,r}$ such that $0 \leq |\mathcal{L}_1(W)| - |\mathcal{L}_2(W)| \leq 1$
- For each $r \geq 3$, if $n_h = h + 2^h$,

$$w(K_{r,r}) = \begin{cases} 2h, & \text{if } n_{h-1} < r < n_h \text{ for some } h \geq 2; \\ 2h+1, & \text{if } r = n_h \text{ for some } h \geq 2. \end{cases}$$

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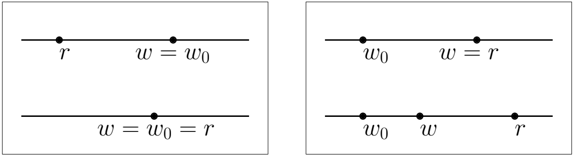
# Feasible values

$$w(K_{r,s}) = w_0(r,s), \text{ if } r \leq w_0(r,s);$$
$$w_0(r,s) \leq w(K_{r,s}) \leq r, \text{ if } r > w_0(r,s).$$




$r \leq w_0$ $\qquad$ $r > w_0$

$$w_0(r,s) \leq w(K_{r,s}) \leq \max\{r, w_0(r,s)\}$$

Given $a, b, c$ with $2 \leq a \leq b \leq c$, find $r, s$, such that $2 \leq r \leq s$ and $w_0(K_{r,s}) = a$, $w(K_{r,s}) = b$, $\max\{r, w_0(r,s)\} = c$

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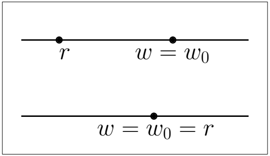
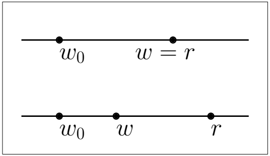



$r \le w_0$

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$$w_0(r,s) \le w(K_{r,s}) \le \max\{r, w_0(r,s)\}$$

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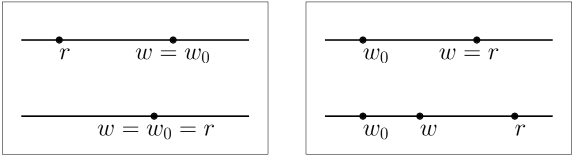


$r \leq w_0$ $\qquad$ $r > w_0$

$$w_0(r,s) \leq w(K_{r,s}) \leq \max\{r, w_0(r,s)\}$$

Given $a, b, c$ with $2 \leq a \leq b \leq c$, find $r$, $s$, such that $2 \leq r \leq s$ and $w_0(K_{r,s}) = a$, $w(K_{r,s}) = b$, $\max\{r, w_0(r,s)\} = c$

# Feasible values

Existence of $r, s$ such that $w_0(K_{r,s}) = a$, $w(K_{r,s}) = b$, and $\max\{r, w_0(r,s)\} = c$:

- If $2 \leq a = b = c$, a solution is $r = a$ and $s = 2^a - a - 1$
- If $2 \leq a = b < c$, there is no solution
- If $2 \leq a < b = c$, there is solution if and only if $a \geq \log_2(2^{c-3} + c + 3)$.
- If $2 \leq a < b < c$, if there is a solution, then $a + \lceil \log_2(c - a + 3) \rceil - 2 \leq b \leq a + \lceil \log_2(c - a + 1) \rceil$

# Watching number of $K_{r,s}$

$w(K_{5,5}) = 4$, and for $s \geq r \geq 3$, not both equal to 5:

$$
\begin{array}{ll}
w(K_{r,s}) = w_0, & \text{if } r \leq w_0; \\
w(K_{r,s}) = w_0 + 1, & \text{if } r = w_0 + 1; \\
w(K_{r,s}) \in \{w_0 + 1, w_0 + 2\}, & \text{if } r = w_0 + 2; \\
w(K_{r,s}) \in \{w_0 + \lceil \log_2(r - w_0 + 1) \rceil, & \\
\qquad w_0 + \lceil \log_2(r - w_0 + 2) \rceil - 1, & \\
\qquad w_0 + \lceil \log_2(r - w_0 + 3) \rceil - 2\} & \text{if } r \geq w_0 + 3.
\end{array}
$$

The identifying number of the complete bipartite graph $K_{r,s}$ is $r + s - 2$ !

# Summary

- Watching systems as an extension of identifying codes
  - Watching systems exist in all graphs
  - $w(G) \leq i(G)$ if $G$ has at least an identifying code
- Watching systems and watching number of complete bipartite graphs
- Open problems
  - Watching number in bipartite graphs and other families
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