A probabilistic approach to consecutive pattern avoiding in permutations

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Consecutive pattern-avoiding permutations – Definition

Let $\pi = (\pi_1 \dots \pi_n) \in \mathcal{S}_n$ *permutation*
and $\sigma = (\sigma_1 \dots \sigma_m) \in S_m$ *pattern*.

We consider $n \gg m$.

Reduction: $st(\pi_{i_1} \dots \pi_{i_k}) = \tau$ if $\tau \in \mathcal{S}_k$ and $\pi_{i_j} < \pi_{i_\ell} \Leftrightarrow \tau_j < \tau_\ell$.

π *contains* the consecutive pattern σ if $\exists\, i$ such that $st(\pi_{i+1} \dots \pi_{i+m}) = \sigma$, otherwise it *avoids* σ.

Example

$n = 5,\ m = 3$:

$\pi = (1\mathbf{542}3)$	reduces to	$st(542) = (321)$
$\pi = (1\mathbf{542}3)$	contains	$\sigma = (321)$
$\pi = (15423)$	avoids	$\sigma = (123)$

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otherwise it *avoids* σ.

For any $\sigma \in \mathcal{S}_m$, we are interested in

$$\alpha_n(\sigma) = |\{\pi \in \mathcal{S}_n : \pi \text{ avoids } \sigma\}|$$

(Elizalde and Noy, 2003)

CMP Conjecture (Elizalde and Noy, 2003)

For any $\sigma \in \mathcal{S}_m$,

$$\alpha_n(\sigma) \le \alpha_n(12\dots m) \, .$$

Theorem (Elizalde, 2006)

For any $\sigma \in \mathcal{S}_m$ the limit

$$\rho_\sigma = \lim_{n\to\infty} \left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n} \qquad (\alpha_n(\sigma) \sim c\rho_\sigma^n n!) \, .$$

CMP Conjecture (Elizalde and Noy, 2003)

For any $\sigma \in \mathcal{S}_m$,

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Consecutive pattern-avoiding permutations – Results

Theorem (Elisalde, 2012+ / P, 2012+)

For any $\sigma \in \mathcal{S}_m$,

$$\rho_\sigma \leq \rho_{(12\ldots m)} \ .$$

Theorem (Elizalde, 2006)

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$$\rho_\sigma = \lim_{n\to\infty} \left(\frac{\alpha_n(\sigma)}{n!}\right)^{1/n} \qquad (\alpha_n(\sigma) \sim c\rho_\sigma^n n!) \ .$$

Fix $\sigma \in \mathcal{S}_m$ and choose $\pi \in \mathcal{S}_n$ uniformly at random.

Define for any $0 \leq i \leq n-m$, the event $A_i = \{st(\pi_{i+1} \dots \pi_{i+m}) = \sigma\}$.

Example

$$\sigma = (123), \pi = (193482576) \Rightarrow \mathbf{A_3} \text{ but } \overline{A_1}$$

Then, the probability of π is σ-avoiding is, $\Pr(\cap_{i=0}^{n-m} \overline{A_i})$.

Thus,

$$\alpha_n(\sigma) = \Pr(\cap_{i=0}^{n-m} \overline{A_i}) n!$$

$$\rho_\sigma = \lim_{n \to \infty} \Pr(\cap_{i=0}^{n-m} \overline{A_i})^{1/n} .$$

Dependencies among events

Fix A_i. For any pattern $\sigma \in \mathcal{S}_m$,

$$\Pr(A_i) = \frac{1}{m!} \;.$$

If they were independent...

$$\rho_\sigma = \lim_{n\to\infty} \Pr(\cap_{i=0}^{n-m}\overline{A_i})^{1/n} = \lim_{n\to\infty} \left(\prod_{i=0}^{n-m} \Pr(\overline{A_i})\right)^{1/n} \sim 1 - \frac{1}{m!} \;.$$

Upper Bound on ρ_σ

Theorem (P, 2012+)

Let $\sigma \in \mathcal{S}_m \backslash \{(12\dots m),(m\dots 21)\}$, then

$$\rho_\sigma \leq 1 - \frac{1}{m!} + O\left(\frac{1}{m^2 \cdot m!}\right) .$$

Suen's inequality, If

$$\mu = \sum \Pr(A_i), \qquad \Delta = \frac{1}{2}\sum_i \sum_{i \sim j} \Pr(A_i \wedge A_j) \quad \text{and} \quad \delta = \max_i \sum_{i \sim j} \Pr(A_j)$$

then,

$$\Pr(\cap \overline{A_i}) \leq \exp\left(-\mu + \Delta e^{2\delta}\right) .$$

We need to take care of $\Pr(A_i \wedge A_j)$: DEPENDS on the pattern.

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Number of permutations with no runs of length m,

$$\rho_{(12\dots m)} \geq 1 - \frac{1}{m!} + O\left(\frac{1}{m \cdot m!}\right) .$$

Theorem (Elisalde, 2012+ / P, 2012+)

CMP conjecture is true.

Theorem (P, 2012+)

Let $\sigma \in \mathcal{S}_m$, then

$$\rho_\sigma \geq 1 - \frac{1}{m!} - O\left(\frac{m-1}{(m!)^2}\right) .$$

One-sided Lovász Local Lemma,

Let H be the dependency graph, if there exists an x such that

$$\Pr(A_i) \leq x(1-x)^{\Delta(H)}$$

then

$$\Pr(\cap \overline{A_i}) \geq (1-x)^n .$$

We just care of $\Delta(H)$: does NOT depend on the pattern.

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TIGHT! $\Longrightarrow (1, 2 \ldots m-2, m, m-1)$

Theorem (P., 2012+)

Let $\sigma \in \mathcal{S}_m$ chosen uniformly at random. For any $1 \leq k < m/2$, we have

$$\rho_\sigma \leq 1 - \frac{1}{m!} + O\left(\frac{4^{m-k}}{(m-k)!m!}\right) ,$$

with probability at least $1 - \frac{2}{(k+1)!} - m2^{-m/2}$.

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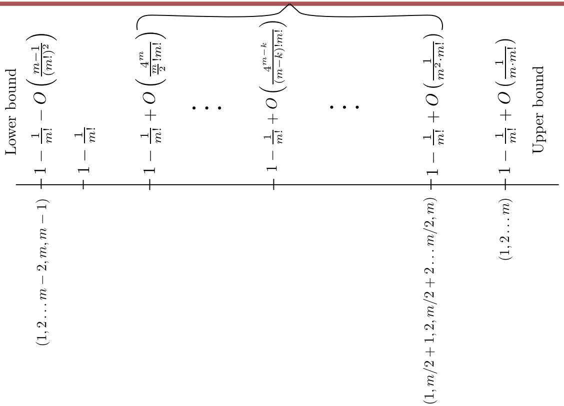

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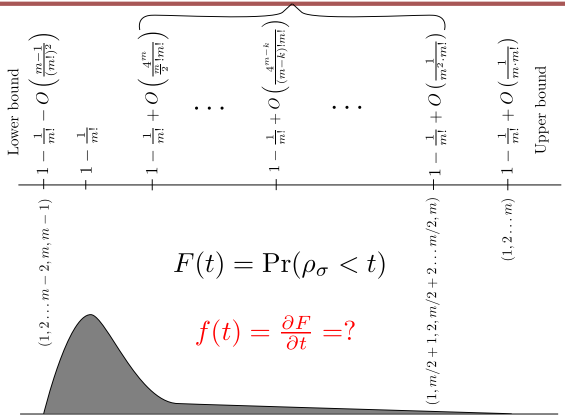
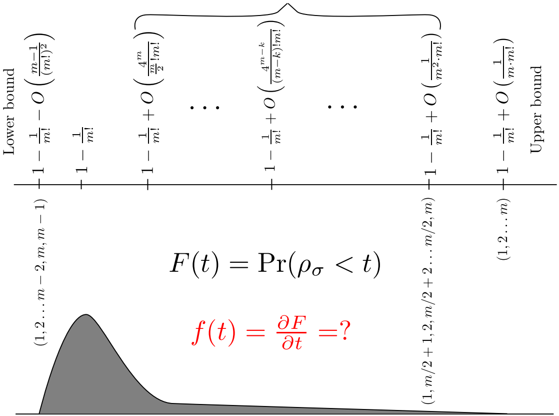


Open question

Let $\sigma \in \mathcal{S}_m$ chosen uniformly at random. Then, we have

$$\rho_\sigma \geq 1 - \frac{1}{m!} + \Omega\left(???\right),$$

with probability at least $\Omega(???)$.



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