Distinguishing Chromatic Numbers of Graphs on Surfaces

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Distinguishing chromatic numbers...

• How many colors do we need to destroy the symmetry of a given graph completely?
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\[ \chi_D(G) \]

- Distinguishing chromatic number
  The minimum number of colors in completely asymmetric colorings.

\[ D(G) \]

- Distinguishing number
  The minimum number of colors in completely asymmetric color assignments

\[ \chi_D(Q_3) = 4, \ D(Q_3) = 3 \]
Graphs that triangulate closed surfaces...

- The complete graphs
  \[ \chi(K_n) = \chi_D(K_n) = D(K_n) = n \]
  \[ \gamma(K_n) = \left\lfloor \frac{(n-3)(n-4)}{12} \right\rfloor \]

- The complete tripartite graphs
  \[ \chi(K_{n,n,n}) = 3, \quad \chi_D(K_{n,n,n}) = 3n \]
  \[ D(K_{n,n,n}) = n + 1 \]
  \[ \gamma(K_{n,n,n}) = \frac{(n-1)(n-2)}{2} \]
Every planar graph is 4-colorable.
• There is no upper bound for the distinguishing chromatic number of 2-connected graphs on any surface.

\[ \chi_D(G) \geq n \]
\[ D(G) \geq n \]

Polyhedral graphs
Two faces meet in at most one vertex or along at most one edge.
Every 3-connected planar graphs is...

- polyhedral.
- **uniquely** and **faithfully** embedded on the sphere.

Any automorphism of a graph can be realized as a symmetric transformation over the sphere.

- Rotation around an axis
- Reflection in the plane, or
- Antipodal map
• Every 3-connected planar graph is 6-distinguishing colorable.

\[ \chi_D(G) \leq \chi(G) + 2 \]
3-Connected planar graphs...

• Theorem (Fijavž, Negami and Sano, 2011)

Every 3-connected planar graph is 5-distinguishing colorable unless it is isomorphic to $K_{2,2,2}$ or $DW_6$.

$$\chi_D(K_{2,2,2}) = 6$$
$$\chi(K_{2,2,2}) = 3$$

$$\chi(DW_6) = 3$$

$$\chi_D(DW_6) = 6$$
• **Theorem (Sano, 2012)**

If a 3-connected planar graph $G$ is isomorphic to none of the followings, then:

$$\chi_D(G) \leq \chi(G) + 1$$

• Exceptions:

$$\overline{K}_2 + C_{2r} \ (r \geq 2), \quad \overline{K}_2 + P_{2k+1} \ (k \geq 1)$$

$$K_1 + C_6, \quad Q_3, \quad R(Q_3), \quad S(Q_3)$$
However, I am a topological graph theorist...
Closed surfaces...

\[ \varepsilon(S_g) = 2 - 2g \]

Sphere \hspace{1cm} Torus \hspace{1cm} Double torus

Projective plane \hspace{1cm} Klein bottle

\[ \varepsilon(N_q) = 2 - q \]
Cup open a closed surface into …
In topological graph theory...

- **CHOICE 1:** Consider only map-automorphisms of embedded graphs for their symmetries.

- **CHOICE 2:** Consider all graph-automorphisms, but use the properties of embedded graphs.
Re-embedding structures of triangulations

- **panel**
  A facial cycle that bounds in all of re-embeddings

- **hole**
  A facial cycle that is not a panel.
Re-embedding structures of triangulations

- Any re-embedding of triangulations with non-empty frame is determined by how the frame is mapped.

**frame**

The subgraph induced by the edges of holes
Re-embeddings and the frame

• If the frame is fixed, then so is the whole

\[ \chi_D(G) \leq \chi_D(\text{Fr}(G)) + \alpha \ldots \]

\[ \chi_D(G) \leq |V(\text{Fr}(G))| + \chi(G) - 3 \]

By Map Color Theorem

Bounded by the maximum number of vertices of locally nonplanar graphs
Faithful embedding
All automorphisms preserve all faces.

**Theorem** The distinguishing chromatic number of a polyhedral graph faithfully embedded on a closed surface does not exceed its chromatic number plus 2 unless it is one of the following exceptions.

\[ \chi_D(G) \leq \chi(G) + 2 \]

By Map Color Theorem

**Exceptions** 3-Colorable triangulations with maximum degree at most 10 \( \chi_D(G) \leq 6 \)
• **Theorem**  Given a closed $F^2$, there exists an upper bound for the distinguishing chromatic numbers of triangulations on $F^2$ of linear order with respect to its genus $g$:

$$\chi^\text{tri}_D (F^2) = O(g)$$

$$\chi^\text{tri}_D (F^2) = O(\sqrt{g}) \ldots ?$$
# Upper bounds for triangulations

<table>
<thead>
<tr>
<th></th>
<th>Sphere</th>
<th>Projective plane</th>
<th>Torus</th>
<th>General</th>
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</thead>
<tbody>
<tr>
<td><strong>Chromatic number</strong></td>
<td>4</td>
<td>6</td>
<td>7</td>
<td>$O(\sqrt{g})$</td>
</tr>
<tr>
<td><strong>Distinguishing number</strong></td>
<td>4 (2)</td>
<td>6 (3)</td>
<td>7 (6)</td>
<td>$O(g)$</td>
</tr>
<tr>
<td><strong>Distinguishing chromatic number</strong></td>
<td>6 (5)</td>
<td>7 (6)</td>
<td>9 (8)</td>
<td>$O(g)$</td>
</tr>
</tbody>
</table>

\[
\chi(K_n) = D(K_n) = \chi_D(K_n) = n
\]

\[
\chi(K_{n,n,n}) = 3, \quad D(K_{n,n,n}) = n + 1, \quad \chi_D(K_{n,n,n}) = 3n
\]
For Spanish friends

- Establish another proof for the theorem on 3-connected planar graphs without Four Color Theorem.
- Find a class of 3-connected planar graphs $G$ with rich symmetry such that $\chi_D(G) = \chi(G)$. 
Thank you for your attention!

Arigatou gozai mashita
Goseichou wo kansha simasu