VIII Jornadas de Matematica Discreta y Algoritmica, Almeria, 11-13 July, 2012

Distinguishing Chromatic Numbers of Graphs on Surfaces

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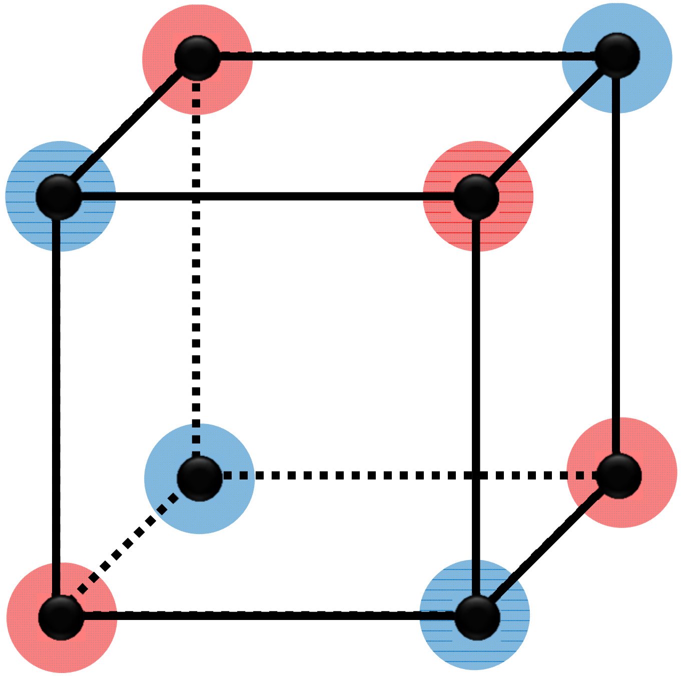
Distinguishing chromatic numbers…

- How many colors do we need to destroy the symmetry of a given graph completely?



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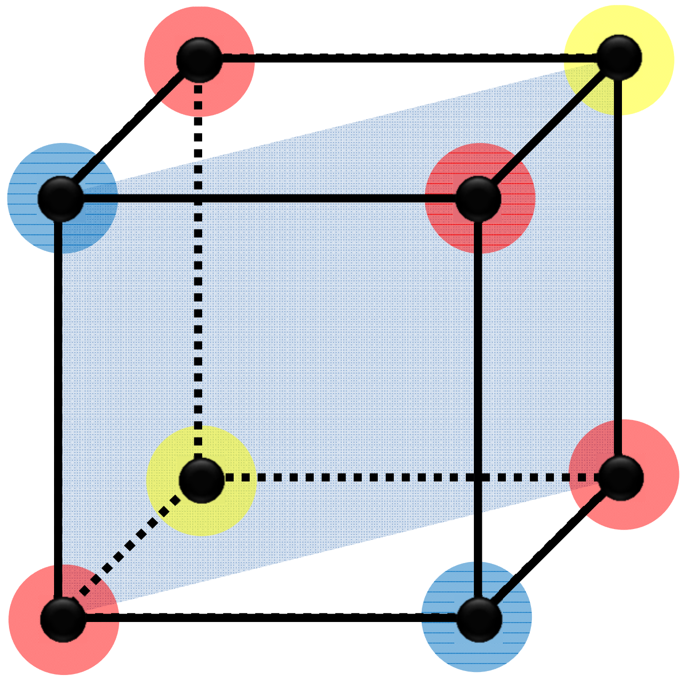

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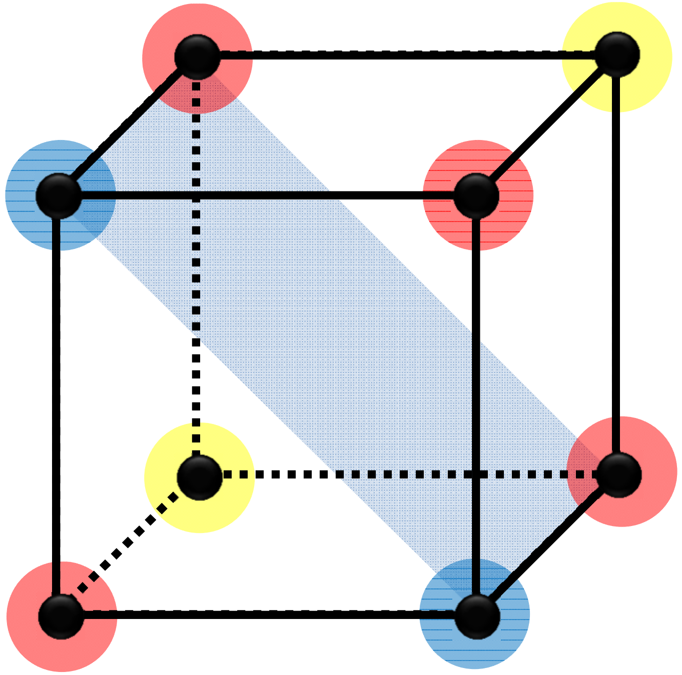


Distinguishing chromatic numbers...

- How many colors do we need to destroy the symmetry of a given graph completely?

$\chi_D(G)$

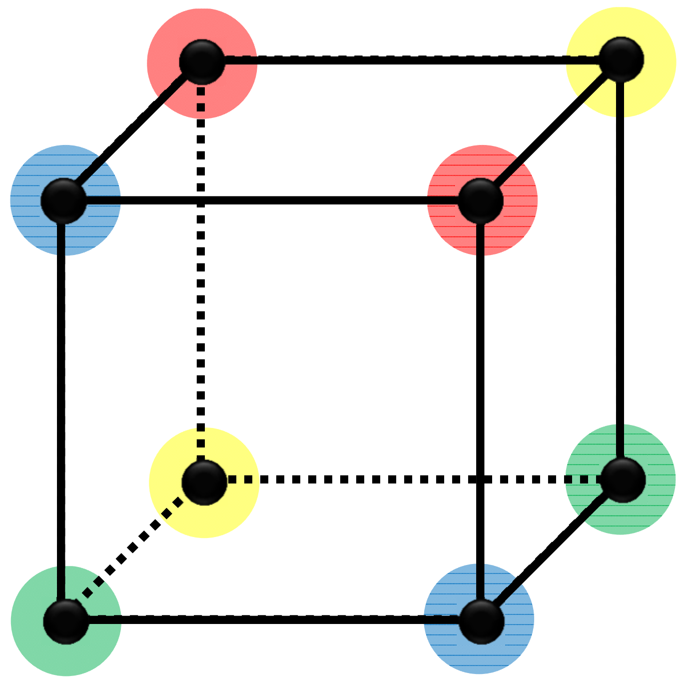
■ Distinguishing chromatic number

The minimum number of colors in completely asymmetric **colorings**.

$D(G)$

■ Distinguishing number

The minimum number of colors in completely asymmetric **color assignments**



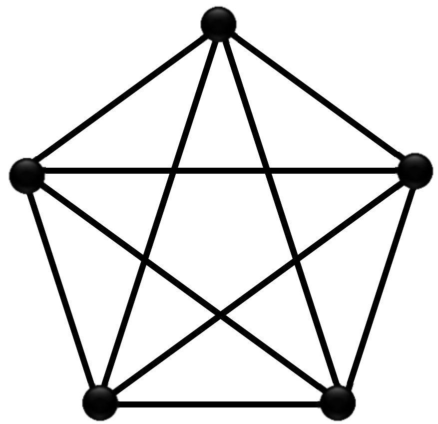
$\chi_D(Q_3) = 4,\ D(Q_3) = 3$

Graphs that triangulate closed surfaces...

- The complete graphs

$$\chi(K_n) = \chi_D(K_n) = D(K_n) = n$$

$$\gamma(K_n) = \left\lceil \frac{(n-3)(n-4)}{12} \right\rceil$$

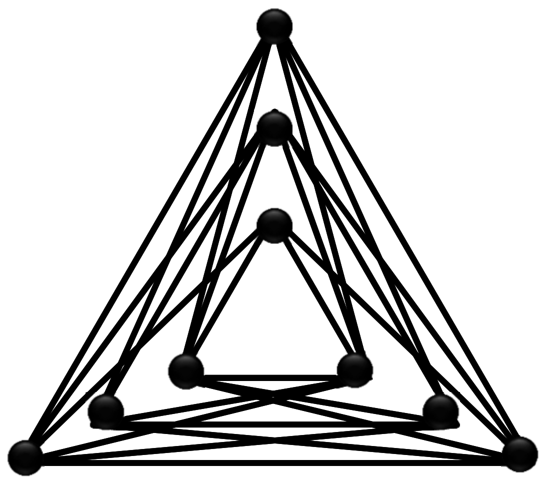


- The complete tripartite graphs

$$\chi(K_{n,n,n}) = 3, \;\; \chi_D(K_{n,n,n}) = 3n$$

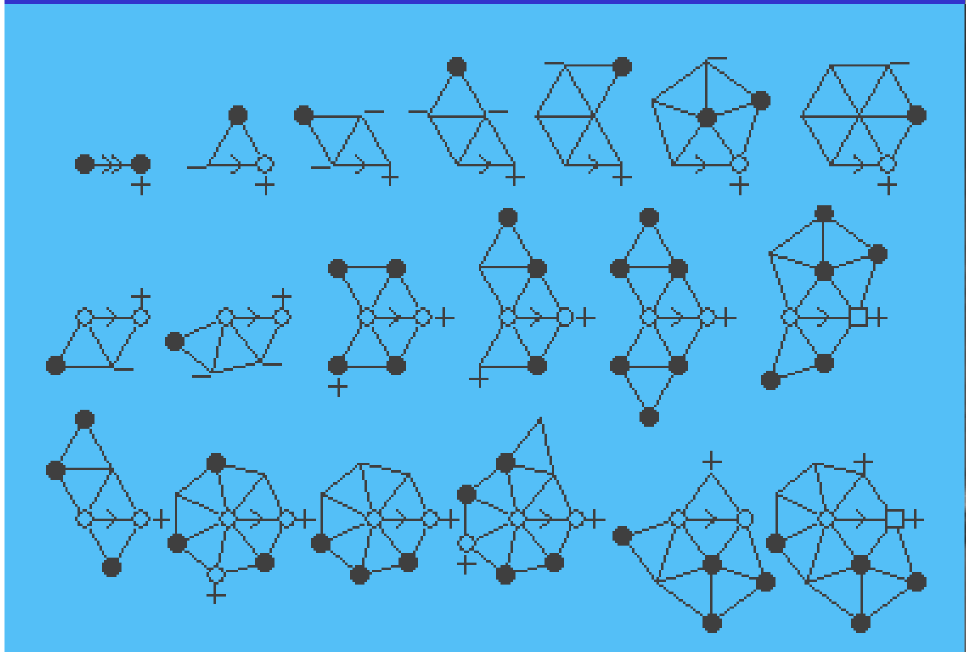
$$D(K_{n,n,n}) = n+1$$

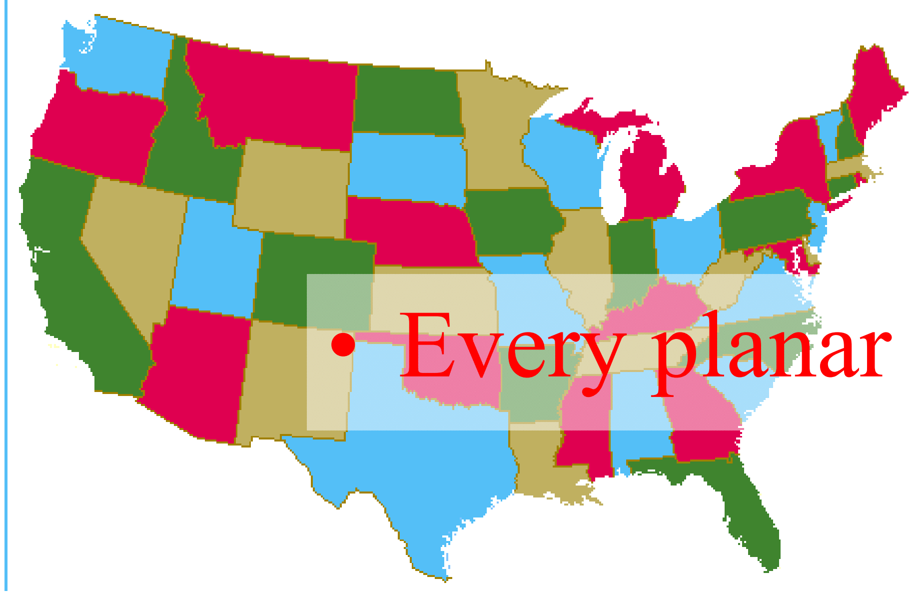
$$\gamma(K_{n,n,n}) = \frac{(n-1)(n-2)}{2}$$

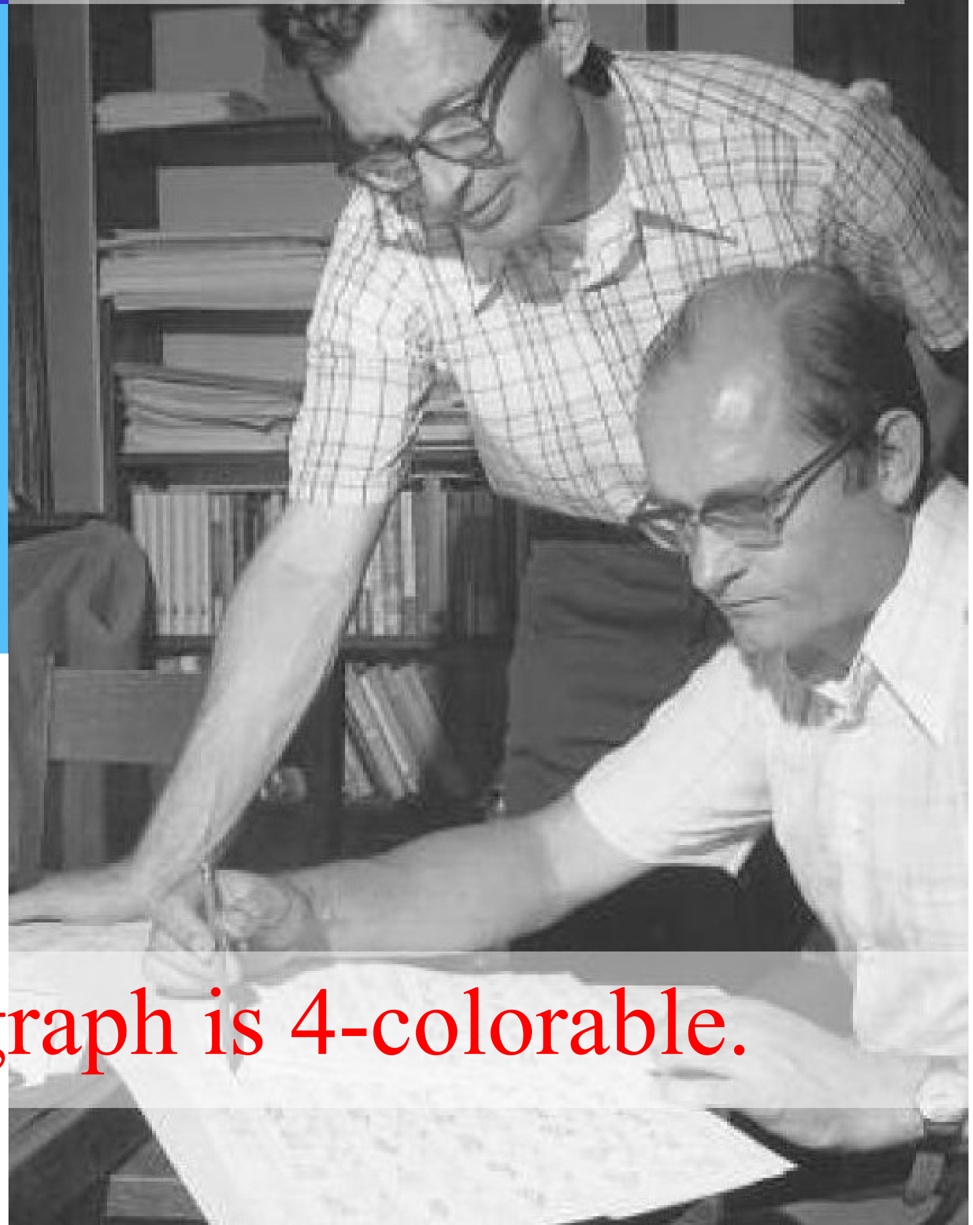


Four Color Theorem

Appel & Haken, 1976





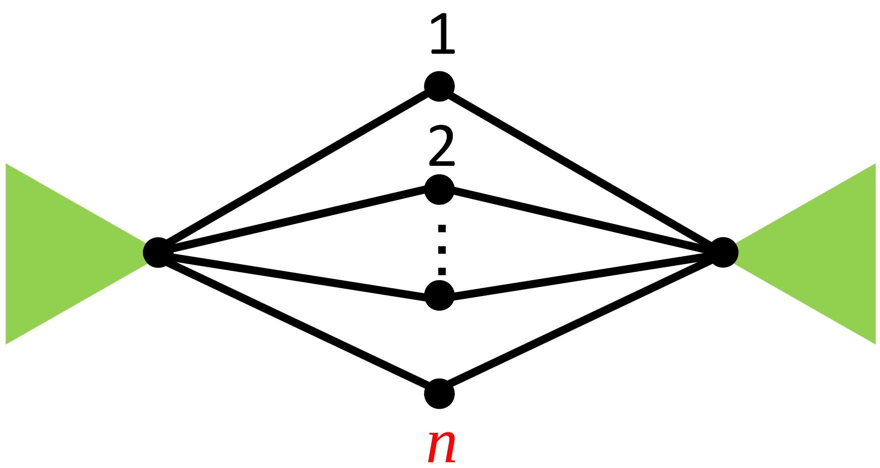


Kenneth Appel and Wolfgang Haken in the 1970s

- Every planar graph is 4-colorable.

No restriction, No upper bound

- There is no upper bound for the distinguishing chromatic number of 2-connected graphs on any surface.



$$\chi_D(G) \geq n$$

$$D(G) \geq n$$

■ Polyhedral graphs

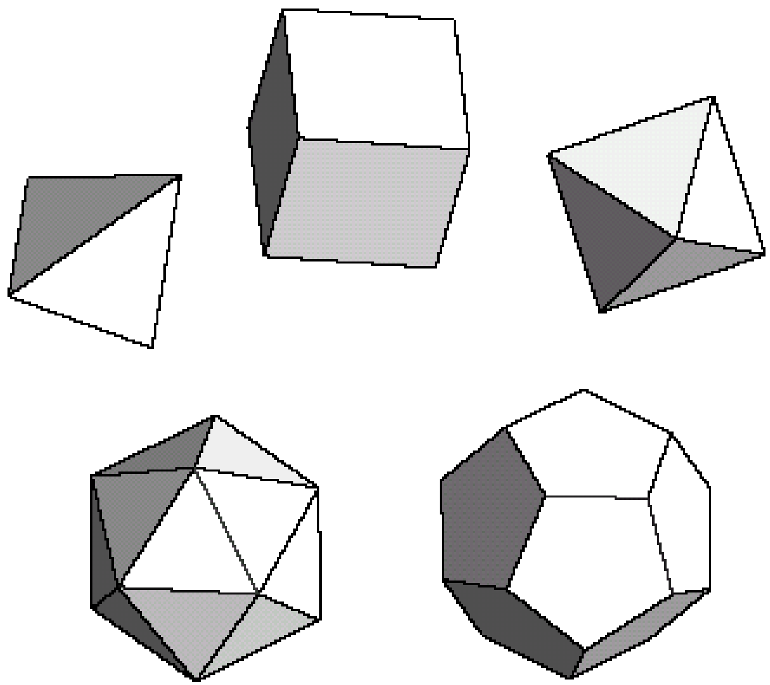
Two faces meet in at most one vertex or along at most one edge.

Every 3-connected planar graphs is…

- polyhedral.
- uniquely and faithfully embedded on the sphere.

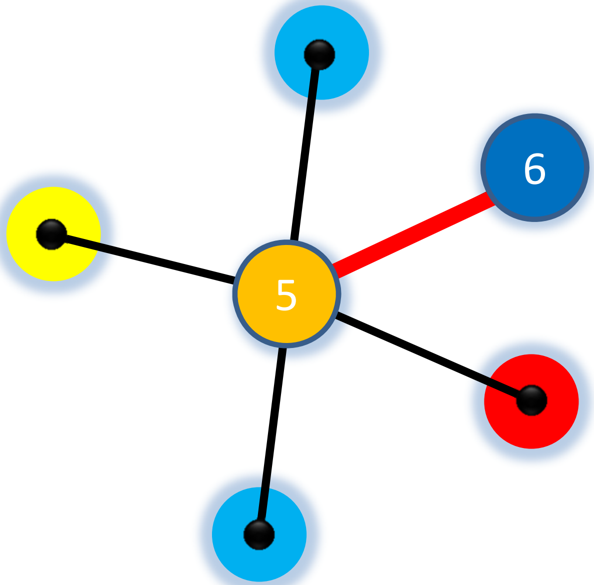
Any automorphism of a graph can be realized as a symmetric transformation over the sphere.

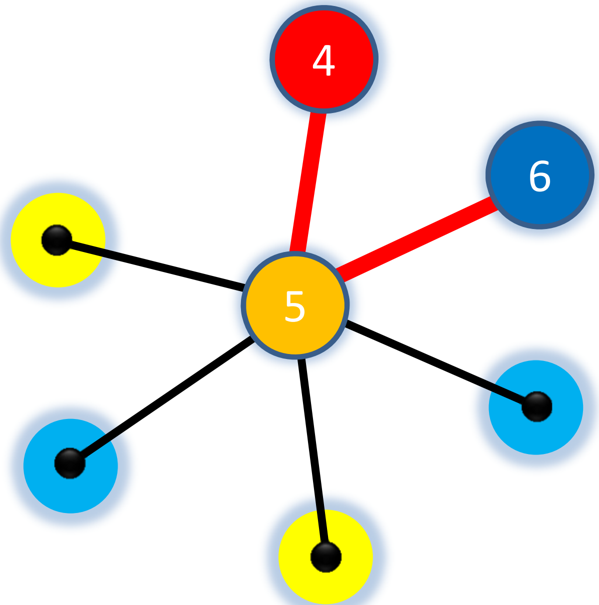
- Rotation around an axis
- Reflection in the plane, or
- Antipodal map



Re-embeddings and the frame

- Every 3-connected planar graph is 6-distinguishing colorable.

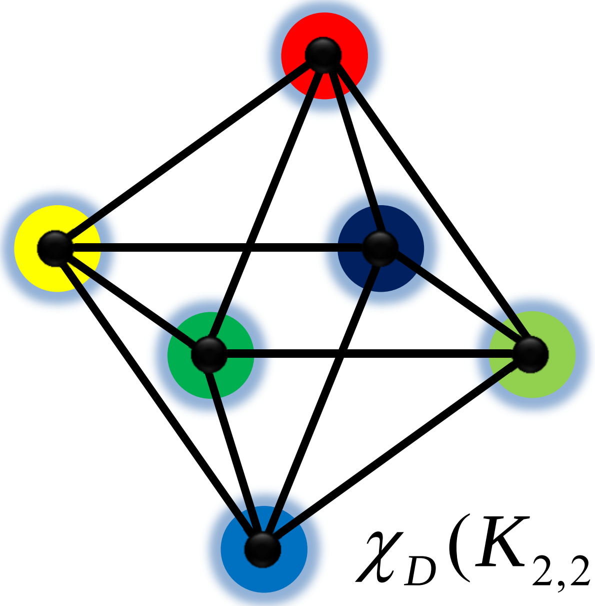




$$\chi_D(G) \leq \chi(G) + 2$$

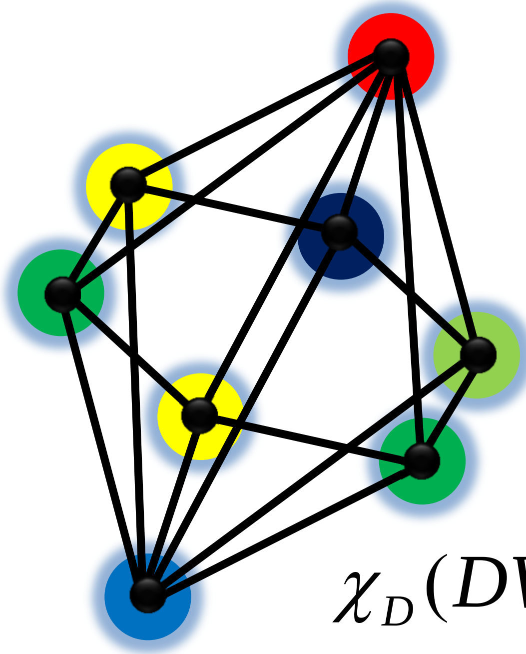
3-Connected planar graphs…

- **Theorem** (Fijavž, Negami and Sano, 2011)
  Every 3-connected planar graph is 5-distinguishing colorable unless it is isomorphic to $K_{2,2,2}$ or DW_6.



$$\chi_D(K_{2,2,2}) = 6$$

$$\chi(K_{2,2,2}) = 3$$



$$\chi(DW_6) = 3$$

$$\chi_D(DW_6) = 6$$

3-Connected planar graphs…

- **Theorem** (Sano, 2012)

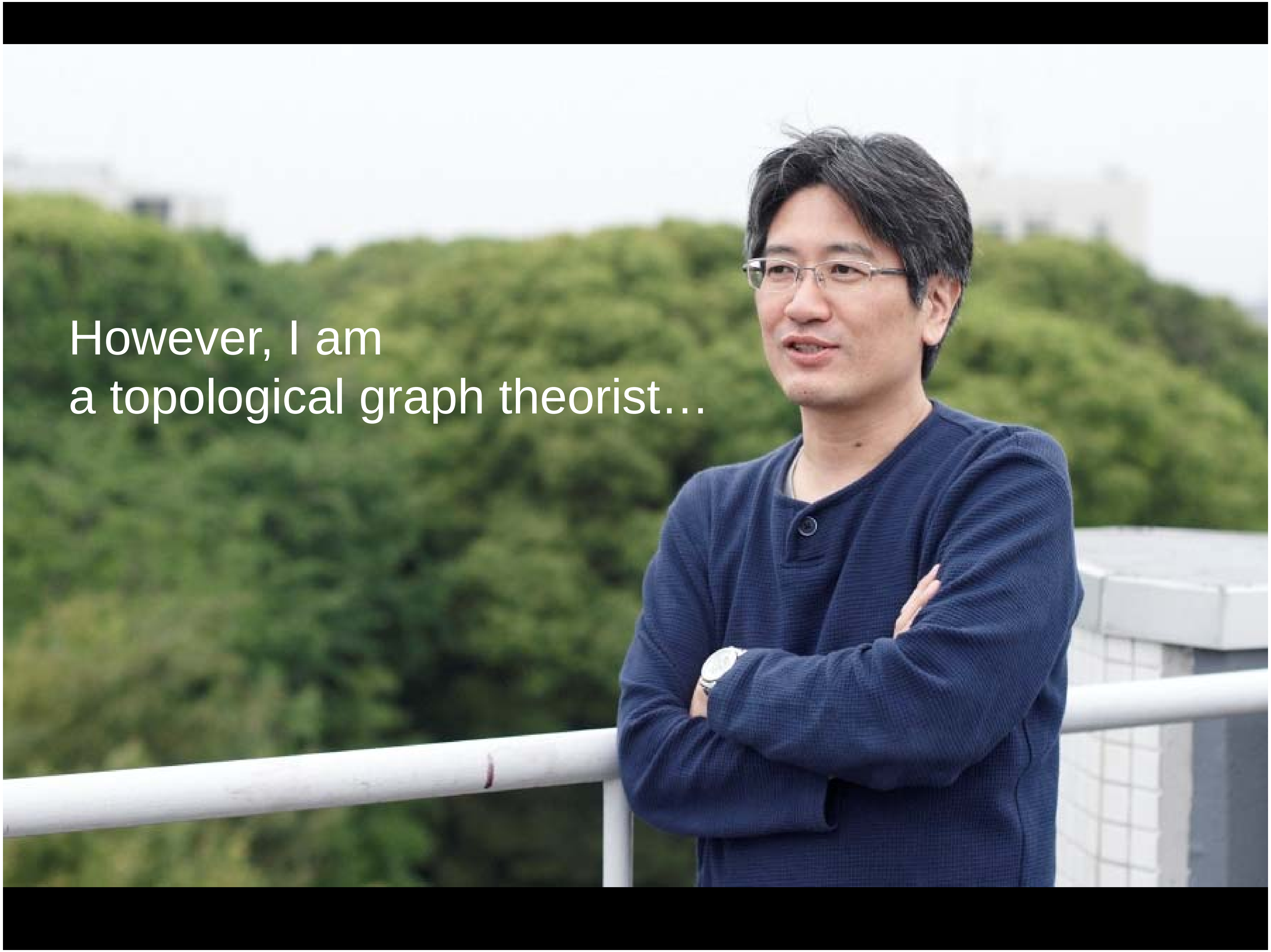
  If a 3-connected planar graph G is isomorphic to none of the followings, then:

$$\chi_D(G) \leq \chi(G) + 1$$

- Exceptions:

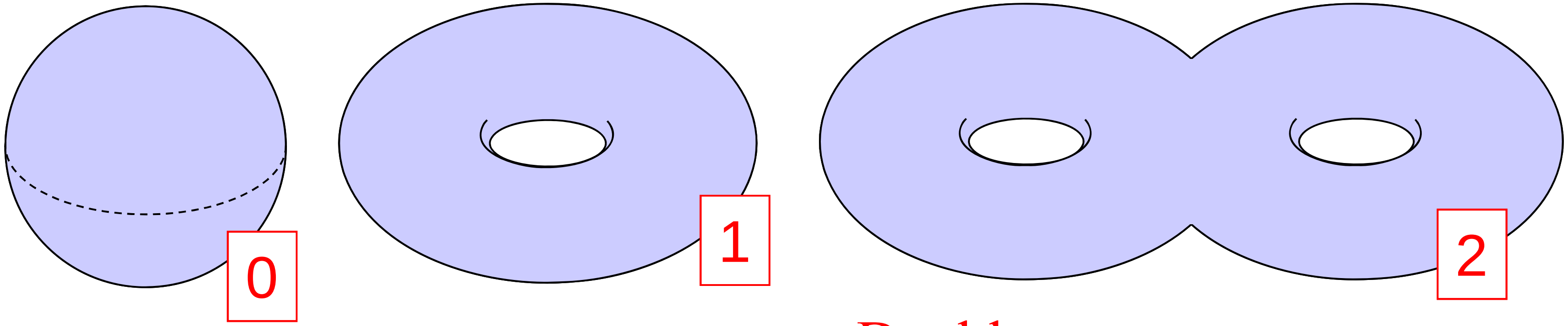
$$\overline{K}_2 + C_{2r}\ (r \geq 2),\quad \overline{K}_2 + P_{2k+1}\ (k \geq 1)$$

$$K_1 + C_6,\ Q_3,\ R(Q_3),\ S(Q_3)$$


However, I am
a topological graph theorist…

Closed surfaces…

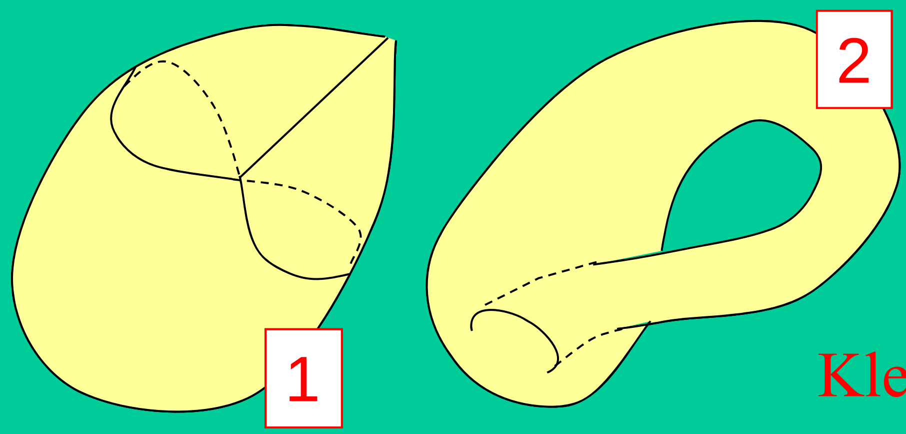
$$\varepsilon(S_g) = 2 - 2g$$



Sphere Torus Double torus

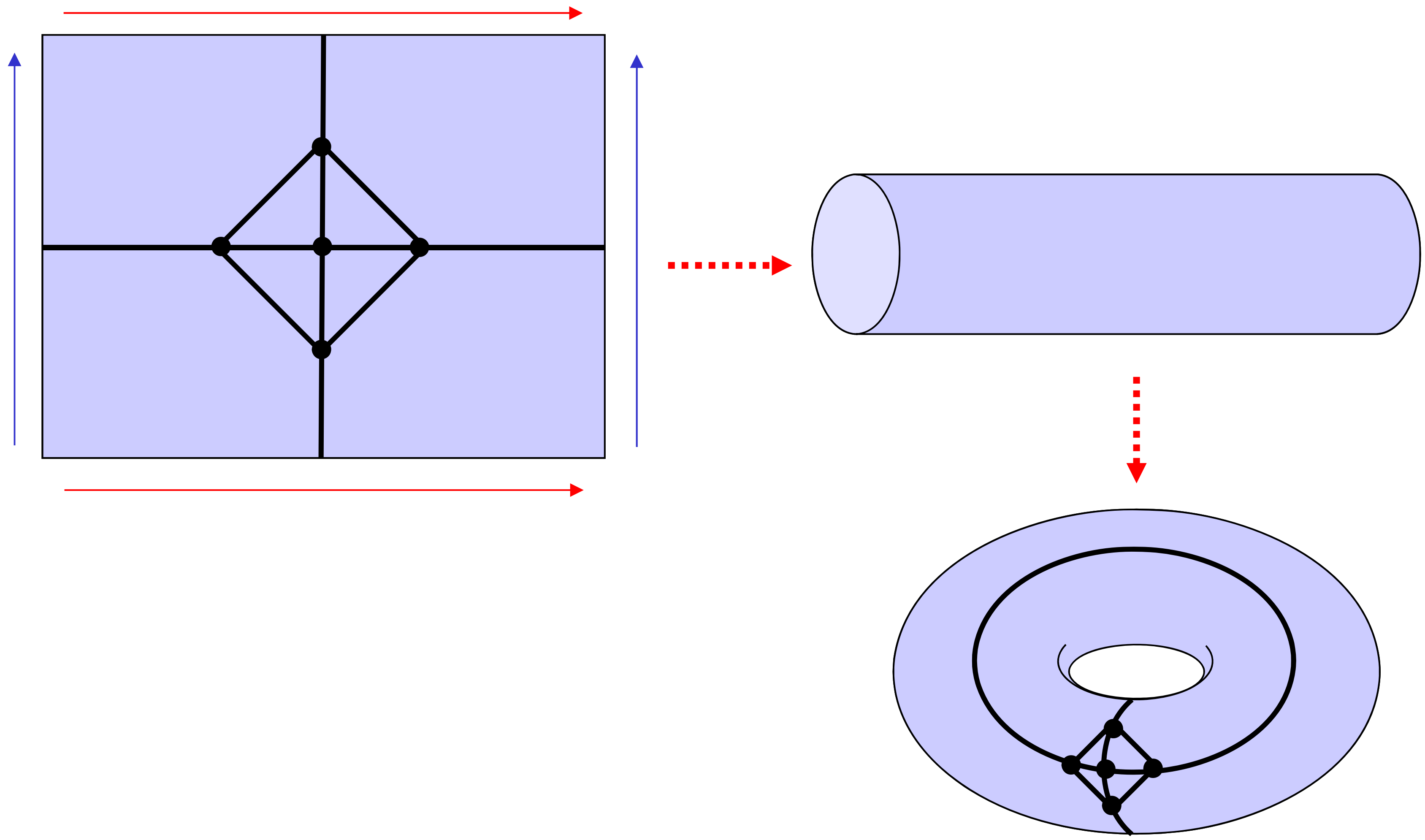
Projective plane

$$\varepsilon(N_q) = 2 - q$$



Klein bottle

Cup open a closed surface into …

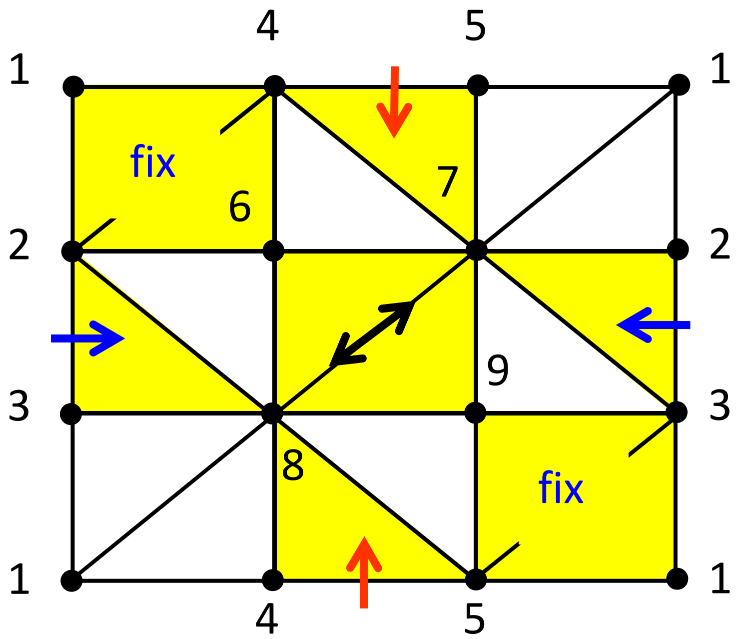


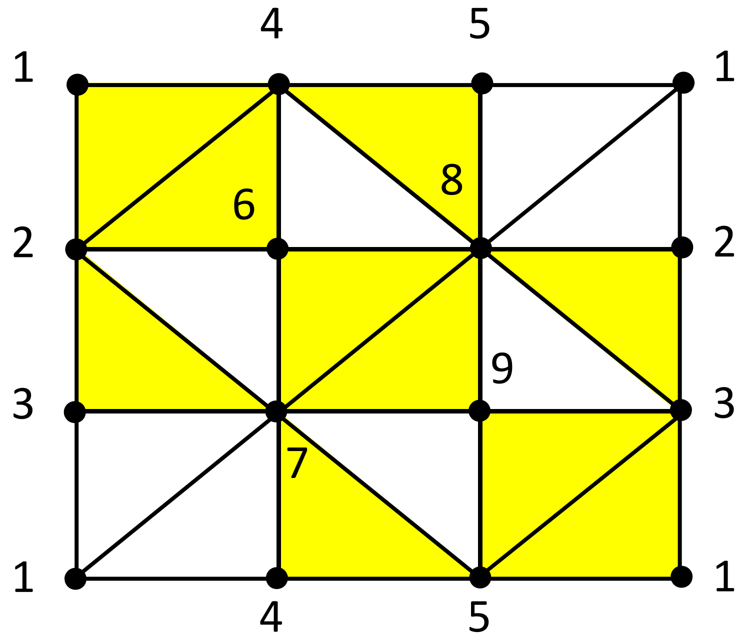
In topological graph theory…

- CHOICE 1:
  Consider only map-automorphisms of embedded graphs for their symmetries.

- CHOICE 2:
  Consider all graph-automorphisms, but use the properties of embedded graphs.

Re-embedding structures of triangulations





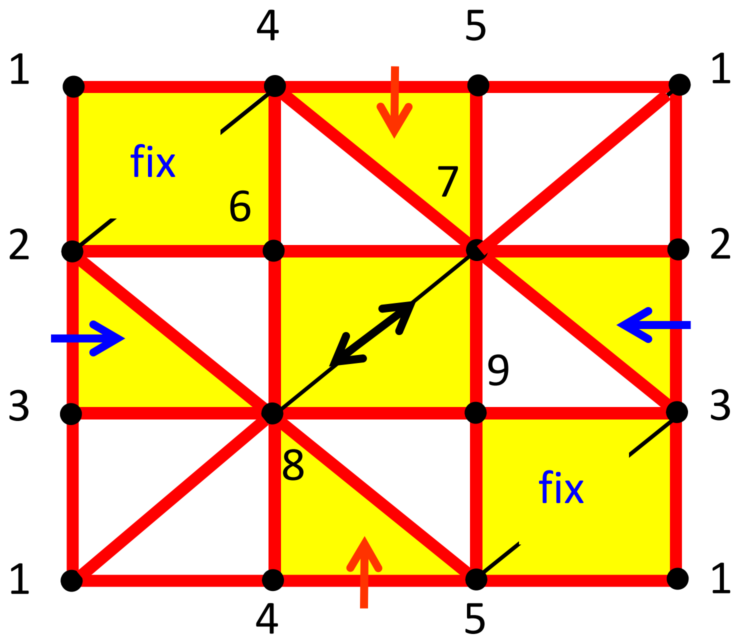
■ panel

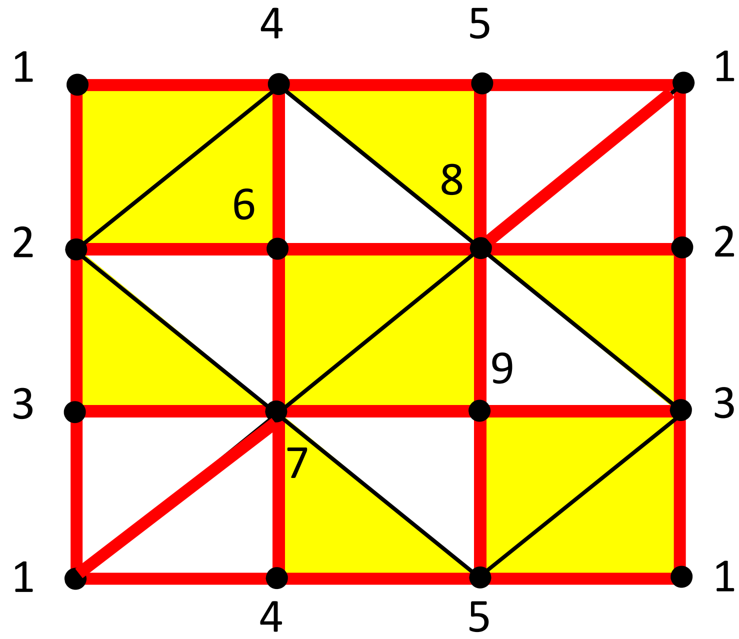
A facial cycle that bounds in all of re-embeddings

■ hole

A facial cycle that is not a panel.

Re-embedding structures of triangulations





■ frame

The subgraph induced by the edges of holes

- Any re-embedding of triangulations with non-empty frame is determined by how the frame is mapped.

Re-embeddings and the frame

- If the frame is fixed, then so is the whole

$$\chi_D(G) \le \chi_D(\mathrm{Fr}(G)) + \alpha \ldots$$

$$\chi_D(G) \le |V(\mathrm{Fr}(G))| + \chi(G) - 3$$

$O(g)$ — Bounded by the maximum number of vertices of locally nonplanar graphs

$O(\sqrt{g})$ — By Map Color Theorem

Being Faithful embedded

■ Faithful embedding

All automorphisms preserve all faces.

$$\mathrm{Fr}(G) = \phi$$

- **Theorem** The distinguishing chromatic number of a polyhedral graph faithfully embedded on a closed surface does not exceed its chromatic number plus 2 unless it is one of the following exceptions.

$$\chi_D(G) \leq \chi(G) + 2$$

$O(\sqrt{g})$ By Map Color Theorem

● **Exceptions** 3-Colorable triangulations with maximum degree at most 10 $\chi_D(G) \leq 6$

Upper bound for triangulations

- **Theorem** Given a closed F^2, there exists an upper bound for the distinguishing chromatic numbers of triangulations on F^2 of linear order with respect to its genus g:

$$\chi_D^{tri}(F^2) = O(g)$$

$$\chi_D^{tri}(F^2) = O(\sqrt{g}) \ldots ?$$

Upper bounds for triangulations

	Sphere	Projective plane	Torus	General
Chromatic number	4	6	7	$O(\sqrt{g})$
Distinguishing number	4 (2)	6 (3)	7 (6)	$O(g)$
Distinguishing chromatic number	6 (5)	7 (6)	9 (8)	$O(g)$

$\frac{7+\sqrt{1+48g}}{2}$

$\frac{9+\sqrt{9+72g}}{2}$

$$\chi(K_n) = D(K_n) = \chi_D(K_n) = n$$

$$\chi(K_{n,n,n}) = 3,\ D(K_{n,n,n}) = n+1,\ \chi_D(K_{n,n,n}) = 3n$$

For Spanish friends

- Establish another proof for the theorem on 3-connected planar graphs without Four Color Theorem.
- Find a class of 3-connected planar graphs G with rich symmetry such that $\chi_D(G) = \chi(G)$.


Thank you for your attention!
Arigatou gozai mashita
Goseichou wo kansha simasu