Grafos radiales de Moore con cintura local máxima

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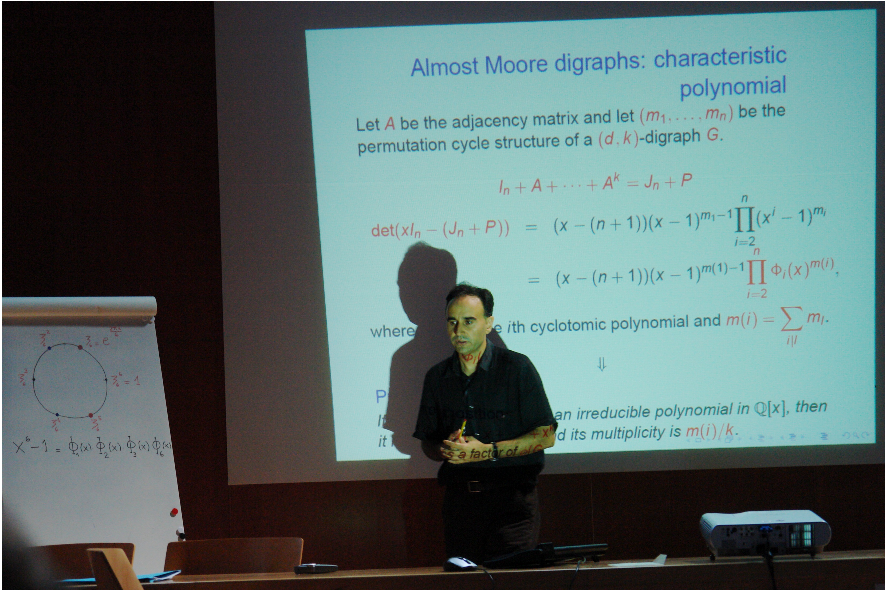
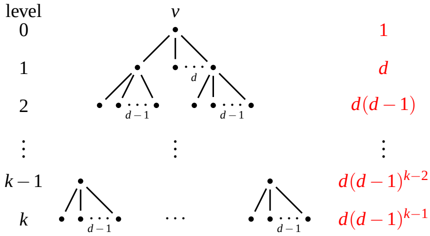

Almost Moore digraphs: characteristic polynomial
Let A be the adjacency matrix and let $(m_1, \ldots, m_n)$ be the permutation cycle structure of a (d, k)-digraph G.
$I_n + A + \cdots + A^k = J_n + P$
$\det(xI_n - (J_n + P)) = (x-(n+1))(x-1)^{m_1-1}\prod_{i=2}^{n}(x^i-1)^{m_i}$
$= (x-(n+1))(x-1)^{m(1)-1}\prod_{i=2}^{n}\Phi_i(x)^{m(i)},$
ith cyclotomic polynomial and $m(i) = \sum_{i|l} m_l$.
an irreducible polynomial in $\mathbb{Q}[x]$, then
d its multiplicity is $m(i)/k$.
$X^6 - 1 = \Phi_1(x)\,\Phi_2(x)\,\Phi_3(x)\,\Phi_6(x)$

Moore graphs

The order of a graph G with maximum degree d and diameter k is at most:



$$M_{d,k} = 1 + d + d(d-1) + \cdots + d(d-1)^{k-2} + d(d-1)^{k-1} \quad \text{(Moore bound)}$$

Definition

A regular graph G of degree d, diameter k and order $M_{d,k}$ is called a Moore graph.

Existence of Moore graphs

Theorem (Banai and Ito ’73, Hoffman and Singleton ’60, Damerell ’73)

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- $k = 2$ *and* $d = 2$ *(Cycle graph* C_5*);*

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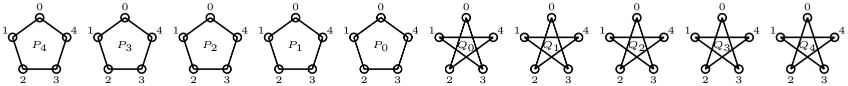
Hoffman-Singleton graph: Vertex i in P_j is joined to vertex $i + jk \pmod 5$ of Q_k



Relaxed Moore graphs

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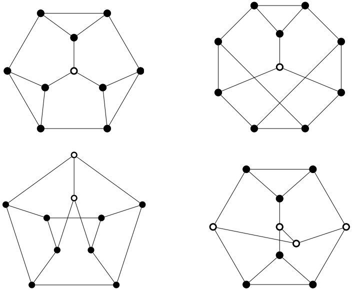
  - **Near-Moore graphs**: graphs on $M_{d,k}$ vertices, diameter k and almost every vertex has degree d (a small set of vertices has degree $d + \delta$).
- diameter-relaxed Moore graphs.
  - **Radial Moore graphs (of defect δ)**: regular graphs of degree d, order $M_{d,k}$, radius k and diameter $k + \delta$.

Radial Moore graphs

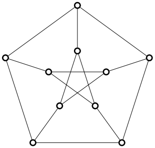
Definition

A regular graph G of degree d, radius k, diameter $\leq k+1$ and order $M_{d,k}=1+d+d(d-1)+\cdots+d(d-1)^{k-1}$ is called a radial Moore graph.

There are 5 non-isomorphic radial Moore graphs for $d=3$ and $k=2$.



Moore graph for $d=3$ and $k=2$



Vertices whose eccentricity is equal to k (minimum possible) are referred to as central vertices.

Existence of radial Moore graphs

The existence of radial Moore graphs is known for the following cases:

- Radius 2 (diameter ≤ 3) and any degree $d \geq 3$.

(Exoo, Gimbert, Gomez and L. 2011)

- Radius 3 (diameter 4) and any degree $d \geq 3$.
- Radius 4 (diameter 5) and degrees $d = 3, 4, 5$.
- Radius 5 (diameter 6) and degree $d = 3$.

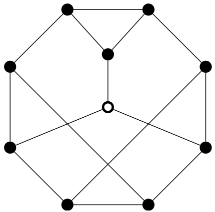
Conjecture

Radial Moore graphs do not exist for a bigger enough radius k and any degree d.

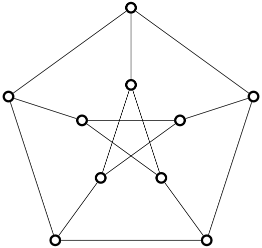
Local girth and the girth vector of a graph

Definition

The *girth* of a vertex v in a graph G, denoted by $g(v)$, is the length of a shortest cycle passing through v. The vector $\mathbf{g}(G)$ constituted by the girths of all its vertices will be referred to as the *girth vector* of G.



$\mathbf{g}(G) : 3^3,\ 4^6,\ 5$



$\mathbf{g}_{3,2} : 5^{10}$

$(\mathbf{g}_{d,k} = 2k + 1^{M_{d,k}})$

How to measure the closeness to a Moore graph

- Every vertex v in a Moore graph of diameter k and degree d has a local girth $g(v) = 2k + 1$, that is, the girth vector of a Moore graph is $\mathbf{g}_{d,k} = (2k + 1, \ldots, 2k + 1)$ of dimension $M(d, k)$.
- Every vertex v in a radial Moore graph of radius k and degree d has a local girth $g(v) \leq 2k + 1$ and the equality holds if v is a central vertex.

Definition

Let $G = (V, E)$ be a radial Moore graph of degree d and radius k. We define

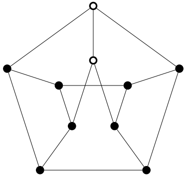
$$\tilde{N}(G) = \|\mathbf{g}(G) - \mathbf{g}_{d,k}\|_1 = \sum_{v \in V} (2k + 1 - g(v)).$$

as a parameter to measure the closeness to a Moore graph.

How to measure the closeness to a Moore graph: Example

- Remember: $\tilde{N}(G) = \|\mathbf{g}(G) - \mathbf{g}_{d,k}\|_1 = \sum_{v \in V}(2k + 1 - g(v))$

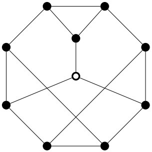
G_1



$\mathbf{g}(G_1) : 4^8,\ 5^2$

$\tilde{N}(G_1) = 8$

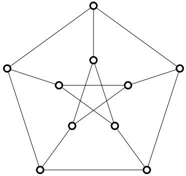
G_2



$\mathbf{g}(G_2) : 3^3,\ 4^6,\ 5$

$\tilde{N}(G_2) = 12$

G_3



$\mathbf{g}(G_3) : 5^{10}$

$\tilde{N}(G_3) = 0$

Definition

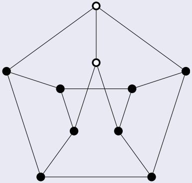
Let G be a radial Moore graph of radius k and degree d. We say that G has *maximum local girth* if $\tilde{N}(G) \leq \tilde{N}(H)$ for all radial Moore graph H.

Maximum local girth graphs

Theorem (Capdevila, Conde, Exoo, Gimbert and L. (2009))

The maximum local girth graphs of degree d and radius k, denoted by $G_{(d,k)}$, *are the following for* $(d,k) \in \{(3,2),(4,2),(3,3)\}$.

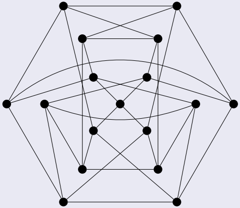
$G_{(3,2)}$



$\mathbf{g}(G_{(3,2)}) : 4^8,\ 5^2$

$\tilde{N}(G_{(3,2)}) = 8$

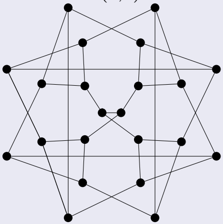
$G_{(4,2)}$



$\mathbf{g}(G_{(4,2)}) : 4^{12},\ 5^5$

$\tilde{N}(G_{(4,2)}) = 12$

$G_{(3,3)}$



$\mathbf{g}(G_{(3,3)}) : 6^{16},\ 7^6$

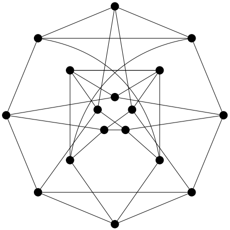
$\tilde{N}(G_{(3,3)}) = 16$

In every case, the digraph is unique.

Table of values of $\tilde{N}(G)$

Theorem (Conde, Exoo, Gimbert and L. (2009))

There exists a radial Moore graph G_d of radius 2 and degree $d > 3$, such that $\mathbf{g}(G_d) : 3^{(d-1)^2},\ 5^{2d}$. As a consequence, $\tilde{N}(G_d) = 2(d-1)^2$.



radius \ degree	d=3	d=4	d=5	$d \geq 6$
$k = 2$	0 (8)	12	32	$2(d-1)^2$
$k = 3$	16	106	259	?
$k = 4$	72	507	1442	?
$k = 5$	272	?	...	...
$k \geq 6$	?	...	...	...

New construction for radius 2 and degree $d \geq 4$.

Proposition

For any degree $d \geq 4$, there exists a radial Moore graph H_d of radius 2 such that $\mathbf{g}(H_d) : 4^{d^2-d}, 5^{d+1}$ and $\tilde{N}(H_d) = d^2 - d$.

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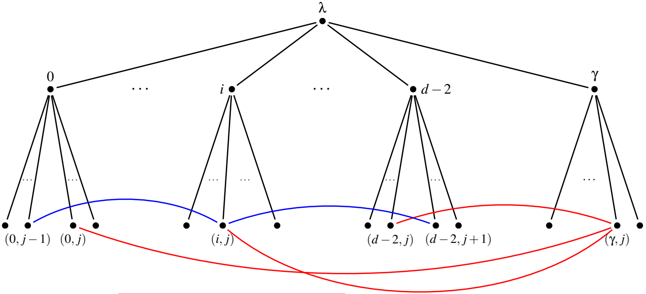

$$(i,j) \longleftrightarrow (\gamma, j) \quad \forall i,j \in \mathbb{Z}_{d-1}$$

$$(i,j) \longleftrightarrow \begin{cases} (i', j+1) \text{ if } i' > i; \\ (i', j-1) \text{ if } i' < i. \end{cases} \quad \text{for } i' \in \mathbb{Z}_{d-1} \setminus \{i\}.$$

Properties of H_d

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- H_4 is the (unique) maximum local girth graph of radius 2 and degree 4.

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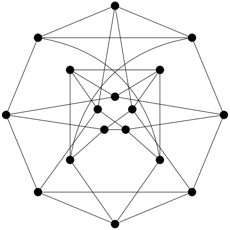
- H_4 is the (unique) maximum local girth graph of radius 2 and degree 4.
- H_d is a minimum of $\tilde{N}$ on the optimization process SA (Simulated Annealing).

Old values of $\tilde{N}(G)$

radius \ degree	d=3	d=4	d=5	$d \geq 6$
$k = 2$	0 (8)	12 [18]	32	$2(d-1)^2$
$k = 3$	16	106	259	?
$k = 4$	72	507	1442	?
$k = 5$	272	?	$\cdots$	$\cdots$
$k \geq 6$	?	$\cdots$	$\cdots$	$\cdots$

- Values given by G_d are depicted in blue color.

G_4

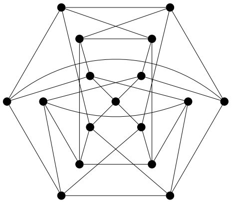


New values of $\tilde{N}(G)$

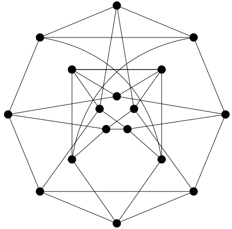
radius \ degree	d=3	d=4	d=5	$d \geq 6$
$k = 2$	0 (8)	12 [18]	25 [32]	$d^2 - d$ $[2(d-1)^2]$
$k = 3$	16	106	259	?
$k = 4$	72	507	1442	?
$k = 5$	272	?	$\cdots$	$\cdots$
$k \geq 6$	?	$\cdots$	$\cdots$	$\cdots$

- Values given by H_d [resp. G_d] are depicted in red [blue] color.

H_4



G_4



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$k\geq 6$	?	$\cdots$	$\cdots$	$\cdots$

Conjecture

H_d is the maximum local girth graph for radius 2 and $d \geq 3$ ($d \neq 7, 57$).

Questions and open problems

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- Prove the uniqueness of maximum local girth graphs.