Constructing almost symmetric numerical semigroups

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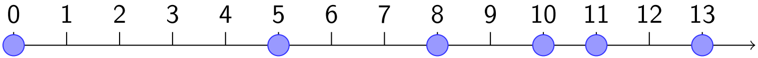
Numerical semigroups

Definition of numerical semigroup

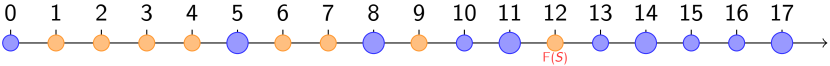
$S \subseteq \mathbb{N}$

- $0 \in S$
- $S + S \subseteq S$
- $\sharp(\mathbb{N} \setminus S) < \infty$ (equivalently $\gcd(S) = 1$)

Example

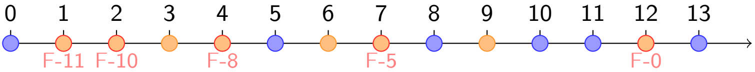


Notable elements



- $S = \langle 5, 8, 11, 14, 17 \rangle = \{a5 + b8 + c11 + d14 + e17 \mid a, b, c, d, e \in \mathbb{N}\}$
- $\mathrm{H}(S) = \mathbb{N} \setminus S = \{1, 2, 3, 4, 6, 7, 9, 12\}$
  $\mathrm{g}(S) = \sharp\mathrm{H}(S) = 8$
- $\mathrm{F}(S) = 12$
- $\mathrm{N}(S) = \{s \in S \mid s < \mathrm{F}(S)\} = \{0, 5, 8, 10, 11\}$
  $\mathrm{n}(S) = \sharp\mathrm{N}(S) = \mathrm{F}(S) + 1 - \mathrm{g}(S) = 5$

Lower bound for the genus



$$\mathrm{F}(S) - \mathrm{N}(S) \subseteq \mathrm{H}(S)$$

$$\mathrm{n}(S) \leq \mathrm{g}(S)$$

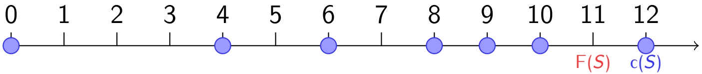
By definition, $\mathrm{n}(S) = \mathrm{F}(S) + 1 - \mathrm{g}(S)$

$$2\mathrm{g}(S) \geq \mathrm{F}(S) + 1$$

Symmetric numerical semigroups

S is symmetric if $\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$

Symmetric numerical semigroups



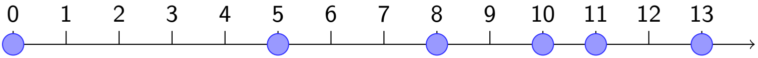
Other characterizations

- $g(S) = n(S)$
- $x \in \mathbb{Z} \setminus S$ implies $F(S) - x \in S$
- $F(S)$ is odd and S is maximal in the set of numerical semigroups with Frobenius number $F(S)$
- $F(S)$ is odd and S is not the intersection of two numerical semigroups containing it

The type of a numerical semigroup

Pseudo-Frobenius numbers

$\mathrm{PF}(S) = \{z \in \mathbb{Z} \setminus S \mid z + S \setminus \{0\} \subseteq S\}$



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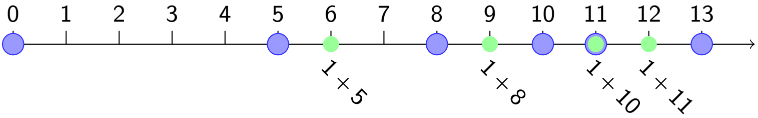

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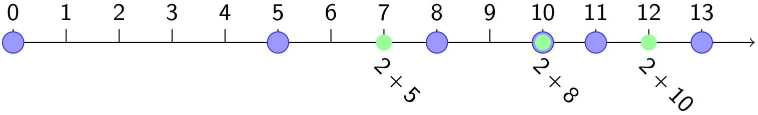

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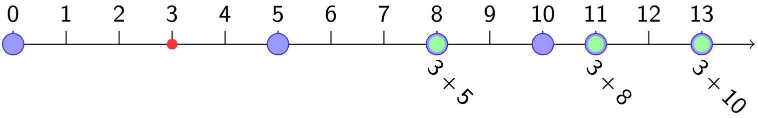

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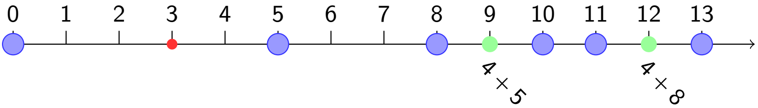

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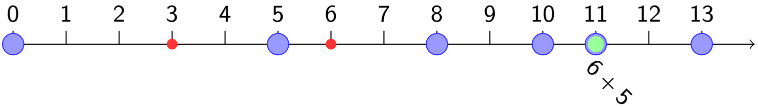

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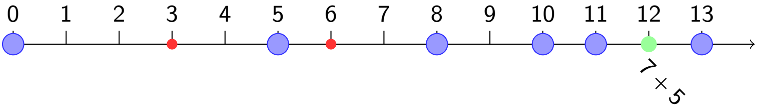

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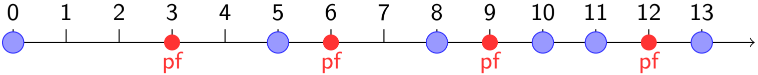

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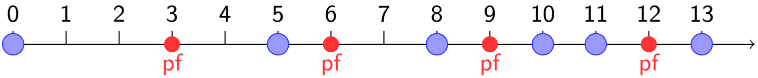

Type

$\mathrm{t}(S) = \sharp\mathrm{PF}(S) = 4$

Irreducible numerical semigroups

S is irreducible if and only if it cannot be expressed as an intersection of two numerical semigroups properly containing it

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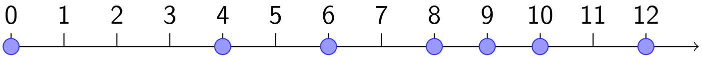
F(S) odd: S irreducible if and only if S is symmetric ($\mathrm{g}(S) = \frac{\mathrm{F}(S)+1}{2}$; $\mathrm{t}(S) = 1$ and $\mathrm{PF}(S) = \{\mathrm{F}(S)\}$)



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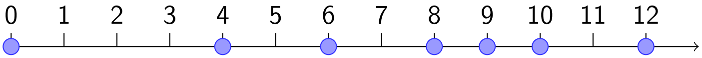
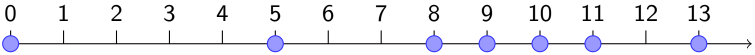


F(S) even: S irreducible if and only if S is pseudo-symmetric ($g(S) = \frac{F(S)+2}{2}$; $t(S) = 2$ and $PF(S) = \{F(S)/2, F(S)\}$)



Thus irreducible implies $g(S) = \frac{F(S)+t(S)}{2}$

Constructing irreducible numerical semigroups

Let f be a positive integer. The set of all irreducible numerical semigroups is a tree routed in

$$\left\{0, \left\lceil \frac{f+1}{2} \right\rceil, \rightarrow \right\} \setminus \{f\}$$

and if S is irreducible, then its sons are $(S \setminus \{x\}) \cup \{f - x\}$ where x is a minimal generator of S such that

(i) $\frac{f}{2} < x < f$ (ii) $2x - f \notin S$ (iii) $3x \neq 2f$ (iv) $4x \neq 3f$

(v) $f - x$ less than the smallest positive integer in S

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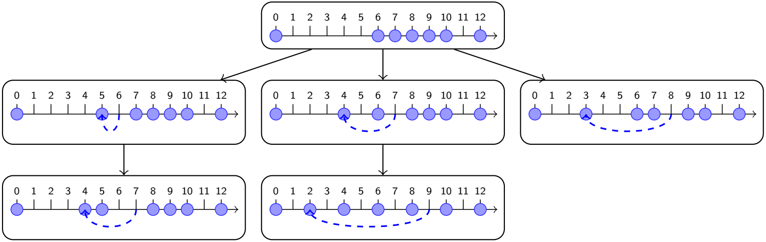
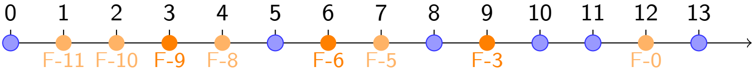


Another lower bound for the genus



$$(\mathrm{F}(S) - \mathrm{N}(S)) \cup (\mathrm{F}(S) - \mathrm{PF}(S) \setminus \{\mathrm{F}(S)\}) \subseteq \mathrm{H}(S)$$

$$2\mathrm{g}(S) \geq \mathrm{F}(S) + \mathrm{t}(S)$$

Almost symmetric numerical semigroups

S is almost symmetric if $\mathrm{g}(S) = \frac{\mathrm{F}(s)+\mathrm{t}(S)}{2}$

Example

Irreducible numerical semigroups are almost symmetric

Almost symmetric numerical semigroups

Barucci-Fröberg-Nari characterizations

- $\mathrm{g}(S) = \frac{\mathrm{F}(S)+\mathrm{t}(S)}{2}$
- $\mathrm{L}(S) = \{x \in \mathrm{H}(S) \mid \mathrm{F}(S) - x \notin S\} \subseteq \mathrm{PF}(S)$
- $\mathrm{PF}(S) = \mathrm{L}(S) \cup \{\mathrm{F}(S)\}$
- $x \in \mathbb{Z} \setminus S$ implies either $\mathrm{F}(S) - x \in S$ or $x \in \mathrm{PF}(S)$
- For every $x \in \mathrm{PF}(S)$, $\mathrm{F}(S) - x \in \mathrm{PF}(S)$ (a similar symmetry on the Apéry sets)

Construction

Constructing almost symmetric from irreducibles

Let T be a numerical semigroup. TFAE

- T is almost symmetric
- $T = S \setminus A$ for S irreducible
  - $\mathrm{F}(S) = \mathrm{F}(T)$
  - A is a subset of the set of minimal generators of S
  - $A \subseteq [\mathrm{F}(S)/2, \mathrm{F}(S)]$
  - for every $x, y \in A$, $x + y - \mathrm{F}(S) \notin T$

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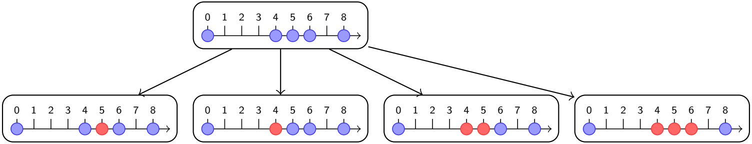

$$\mathrm{PF}(T) = \mathrm{PF}(S) \cup A \cup (\mathrm{F}(S) - A)$$