A new kind of irreducible triangulations of the Möbius band

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Preliminaries

A *punctured surface* F^2 is a compact surface with a hole obtained from a closed compact connected surface by the deletion of the interior of a disk (hole).

The **disk** is the punctured **sphere**

The **Möbius band** is the punctured **projective plane**

A *triangulation* T on a surface F^2 is a simple graph T embedded in F^2 so that each face is bounded by a 3-cycle and any two faces share at most one edge.

The problem

How can we define the notion of "minimal" triangulation?

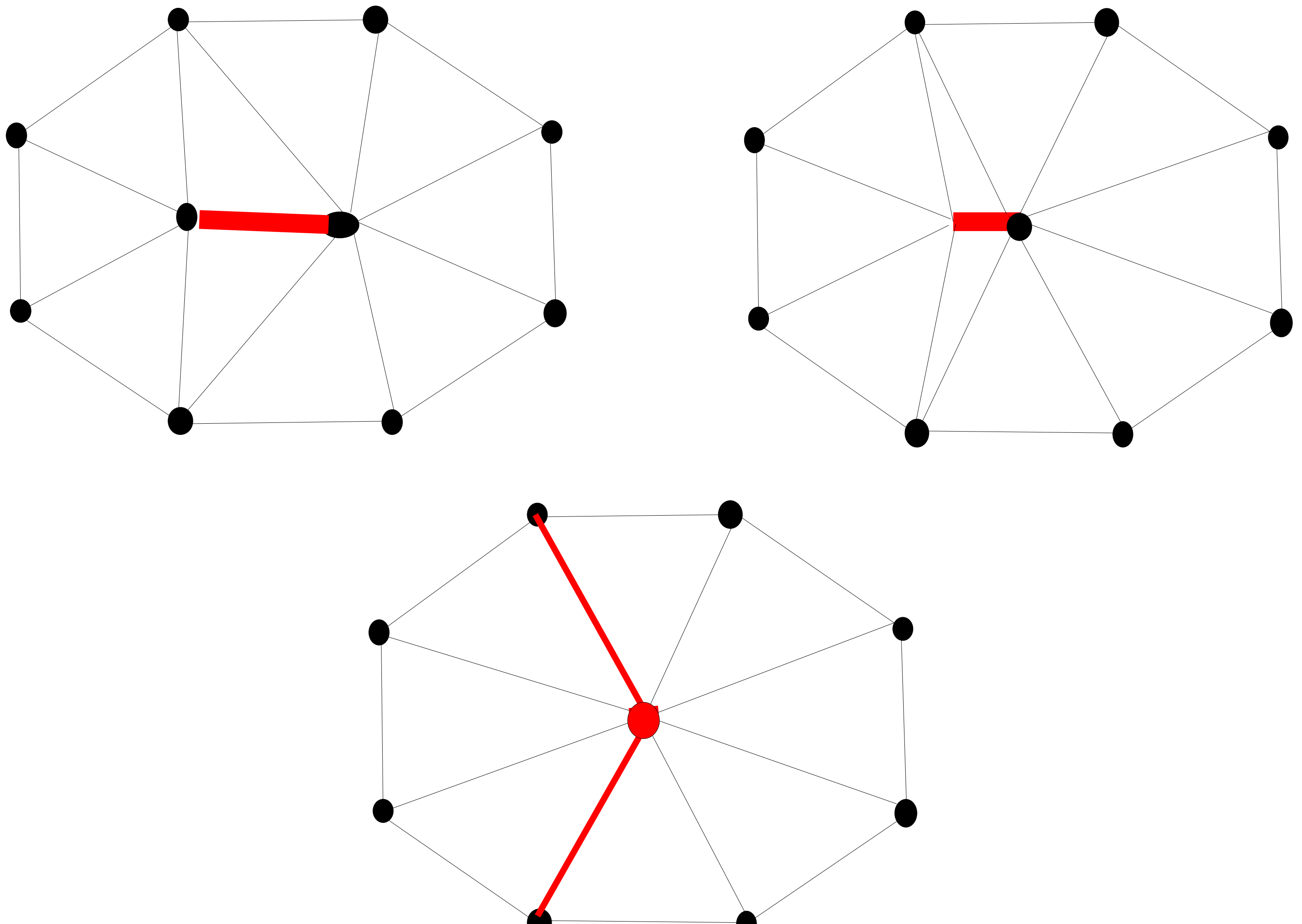
Given a closed surface F^2, is there a set ***T*** of triangulations of F^2, so that any other triangulation of F^2 can be obtained from a triangulation of ***T*** by applying an "easy" method?

- Sphere, Steinitz-Rademacher (1934)
- Projective plane, Barnette (1982)
- Torus, Lawrencenko (1987)
- Klein Bottle, Lawrencenko-Negami (1997), Sulanke (2006)

What about surfaces with boundary?

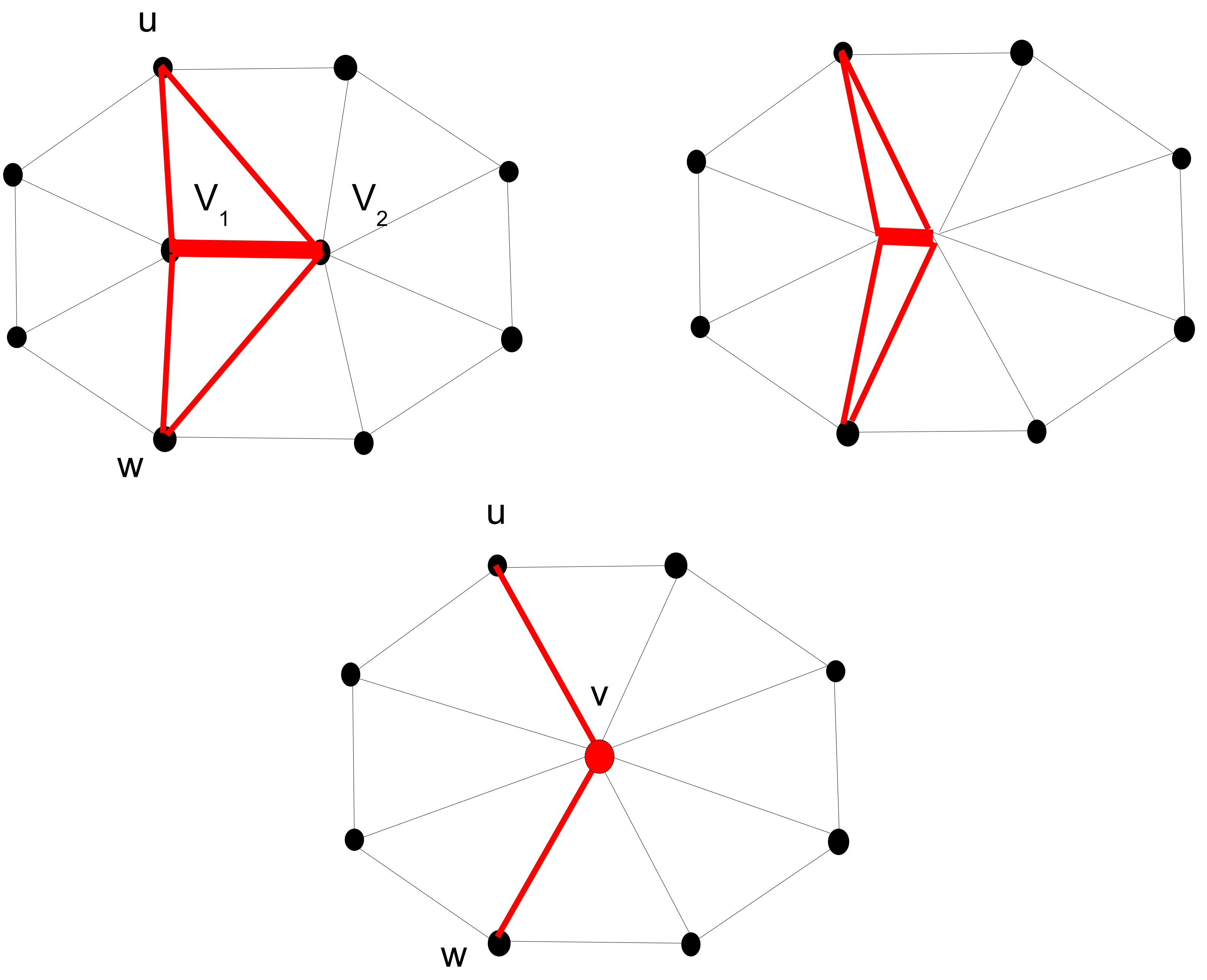
Operations on graphs

Edge contraction



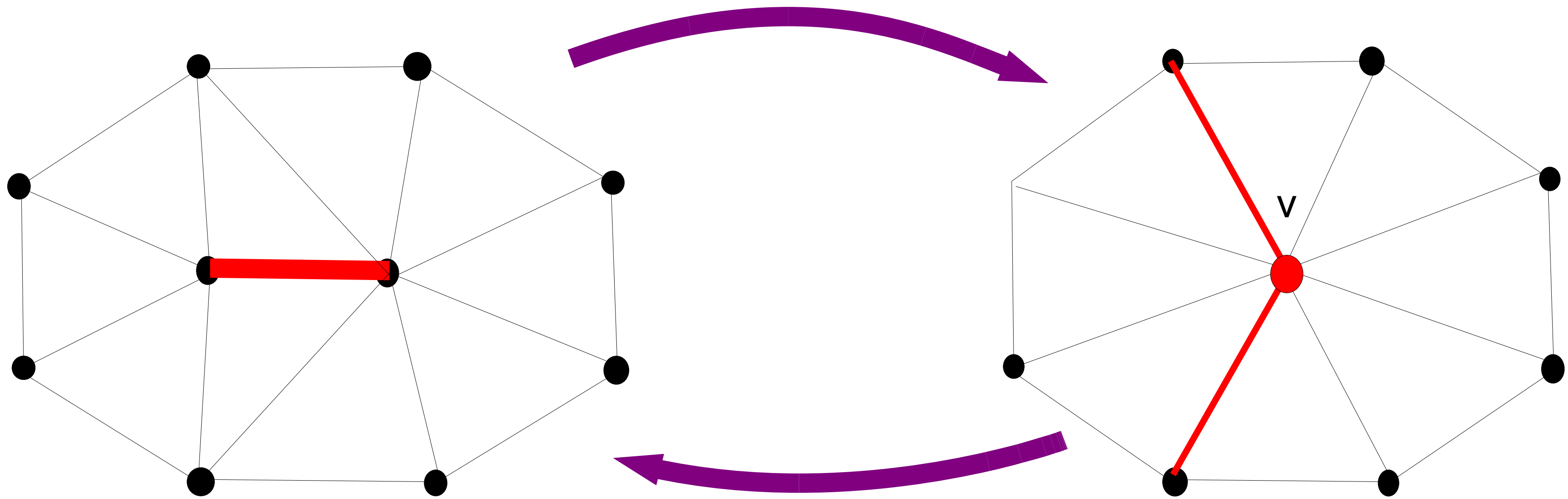
Operations on graphs

Vertex splitting



Operations on graphs

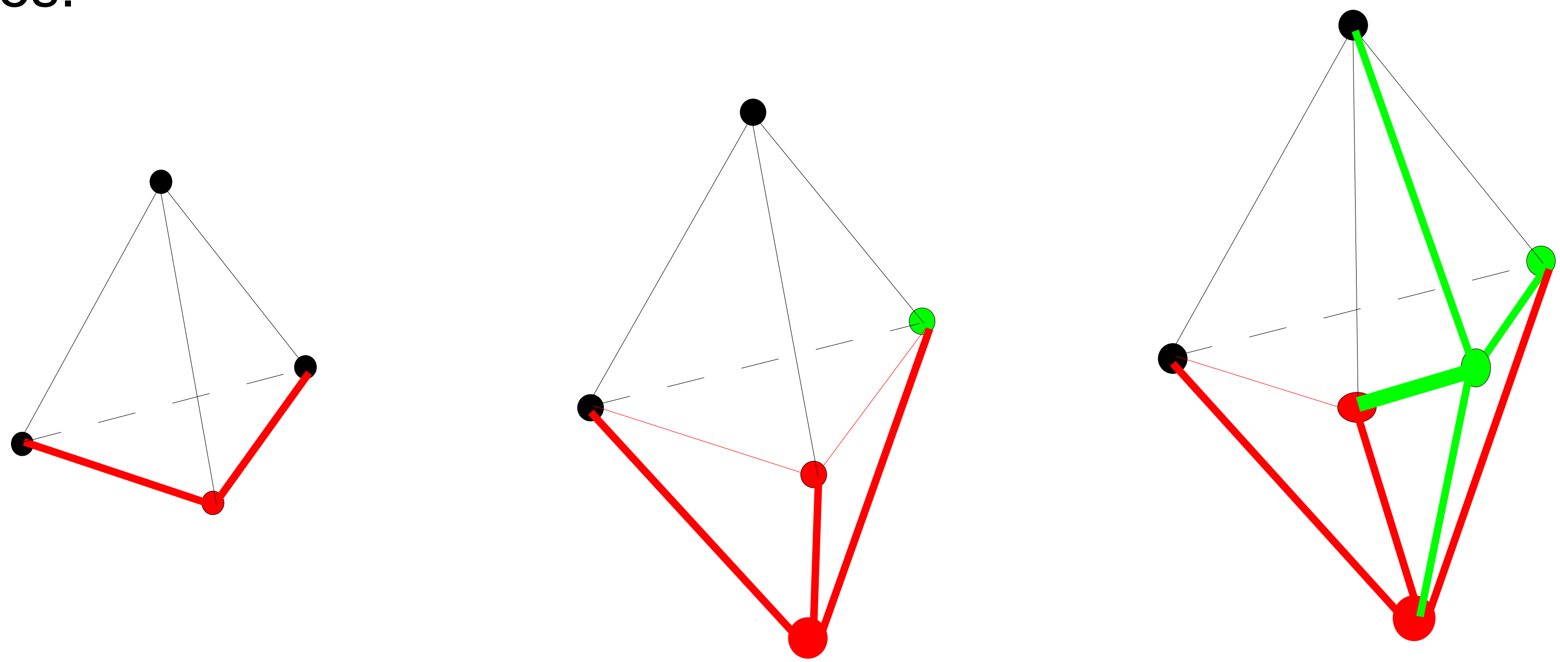
Edge contraction



Vertex splitting

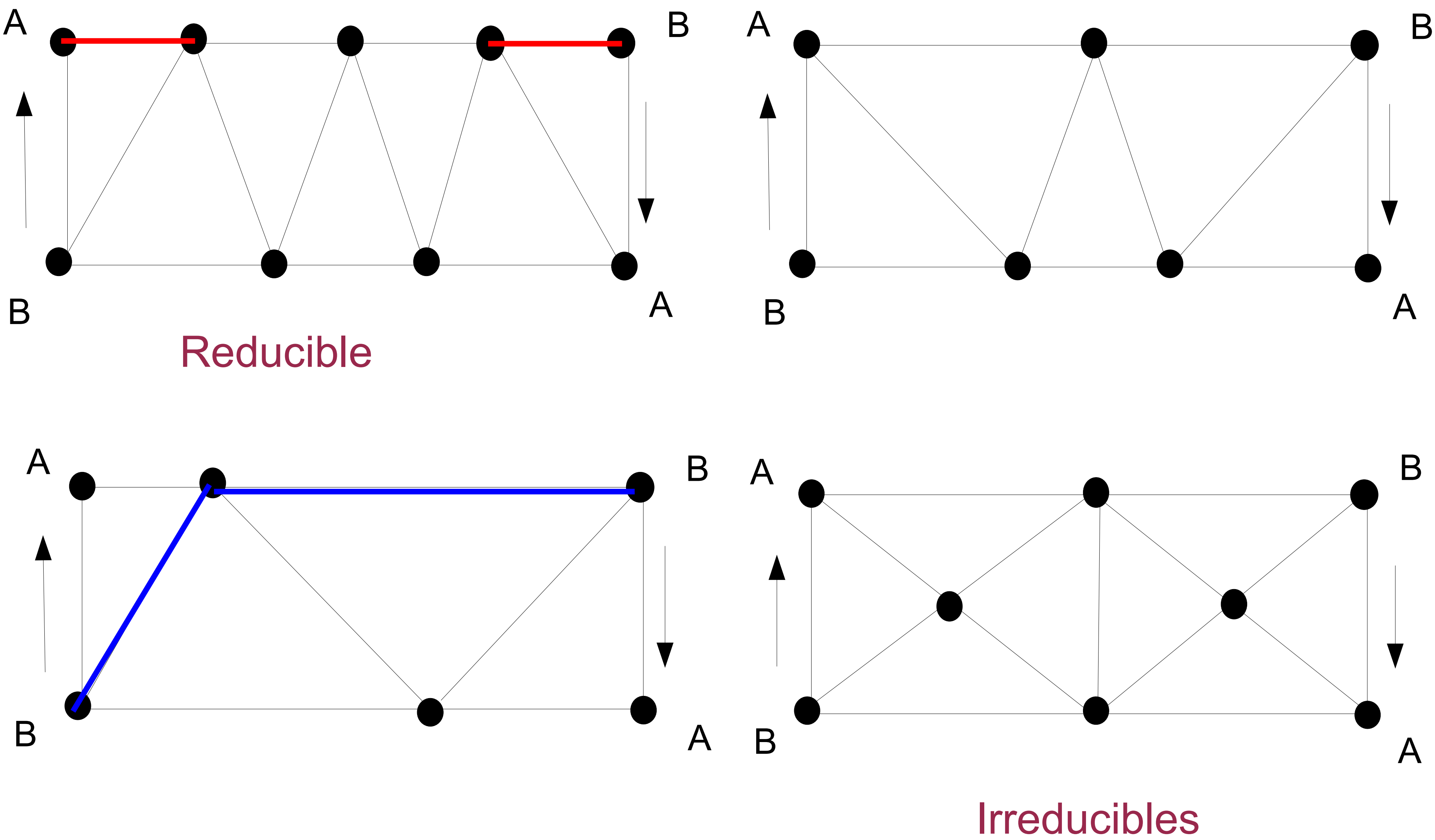
Irreducible triangulations

- Let T be a triangulation of a surface F^2. An edge **e** of T is said to be *contractible* if *the* graph obtained after contracting **e** in T is a triangulation of the same surface F^2.
- A triangulation T is said to be *irreducible* if it has no contractible edges.



Steinitz, 1934: the tetrahedron is the only irreducible triangulation for the sphere.

Irreducible triangulations of the Möbius band

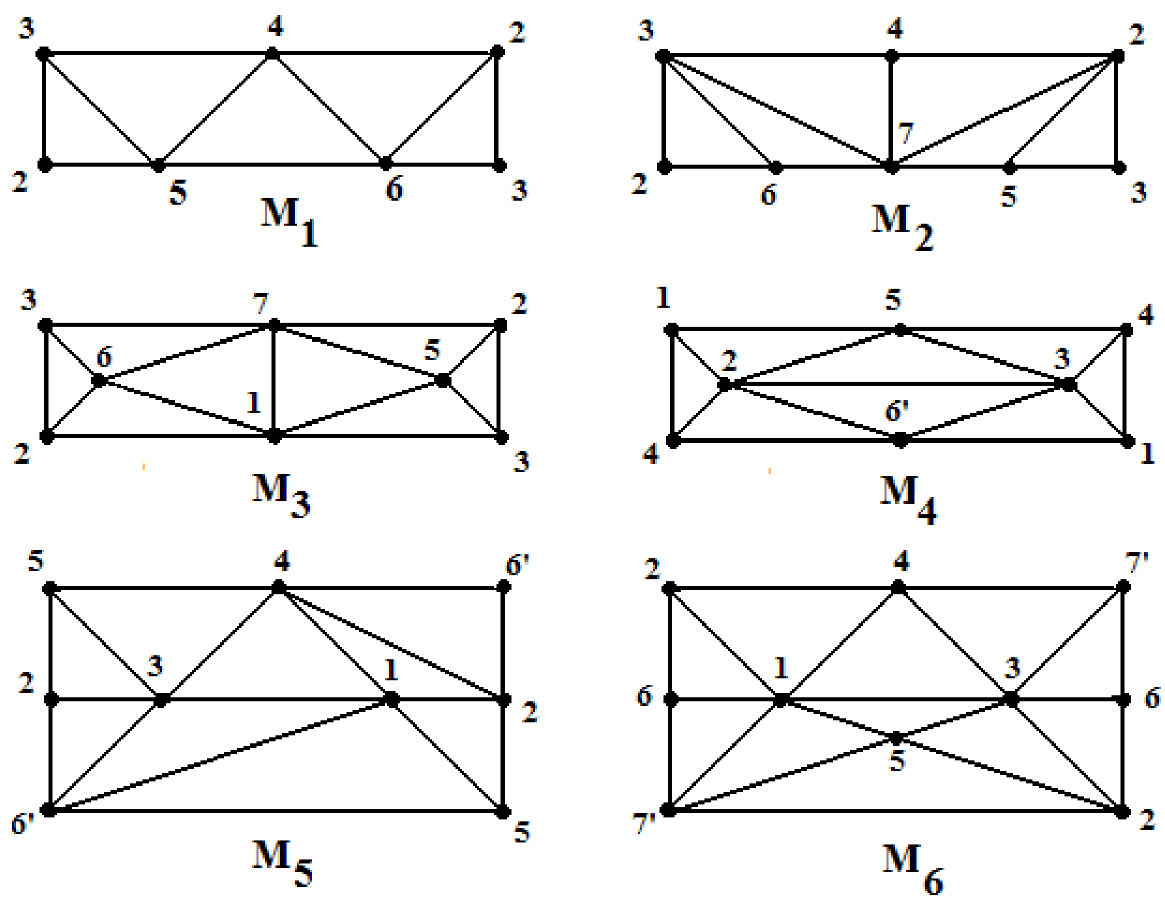


Irreducible triangulations of the Möbius band

Theorem (Chávez, Quintero, Lawrencenko and Villar, 2012)

For any surface ***with boundary****, there is a finite number of irreducible triangulations.*

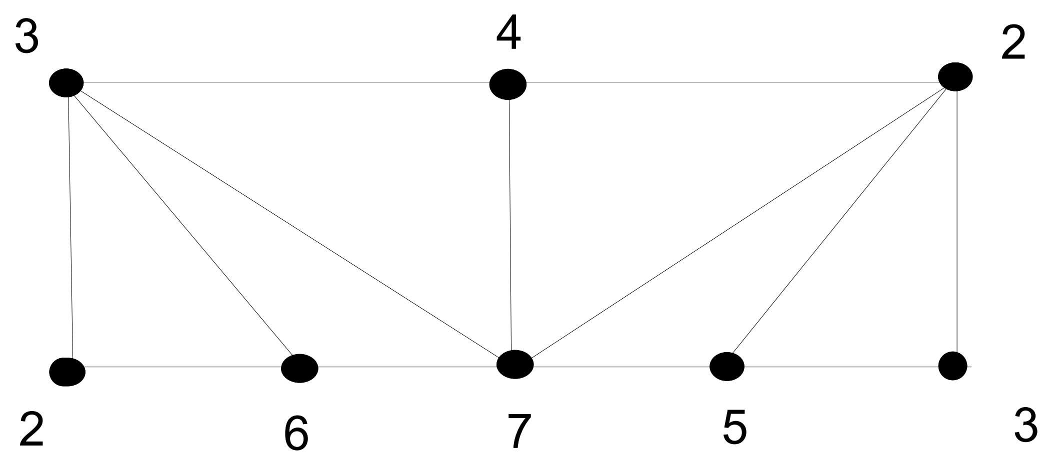
There are six irreducible triangulations of the Möbius band:



(Θ; 4)-Irreducible triangulations of punctured surfaces

- Let *T* be a triangulation with minimun degree ≥ 4, an edge ***e*** of *T* is said to be *4-contractible* if its contraction keeps the degree ≥ 4

- Any irreducible triangulation of an arbitrary non spherical closed surface F^2 has minimun degree ≥ 4.



(Θ, 4)-Irreducible triangulations of punctured surfaces

Theorem (Nakamoto and Negami, 2002)

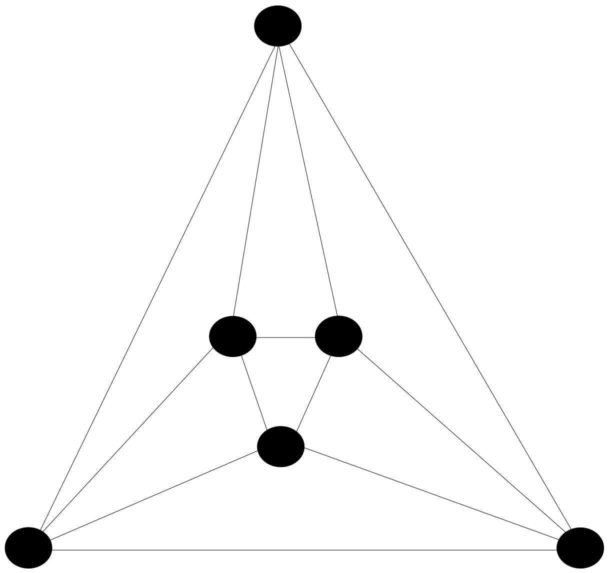
Any triangulation on a closed surface F^2, except the sphere, with minimum degree at least 4, can be obtained from an irreducible triangulation by a sequence of 4-splitting and addition of octahedra.

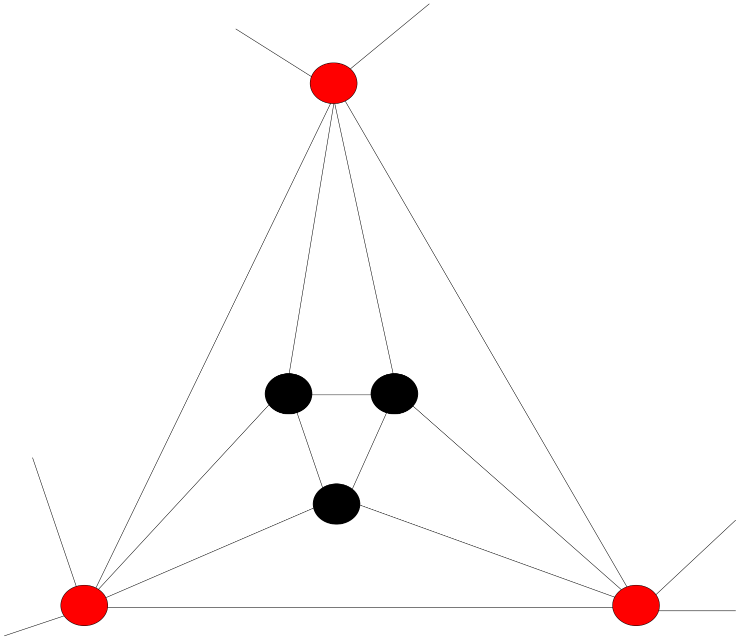
Problem:

How can we prove an analogous theorem for surfaces with boundary?

(Θ, 4)-Irreducible triangulations of punctured surfaces

Addition of an octahedron / Removing an octahedron





It is a particular case of face subdivision.

It is equivalent to three consecutive splittings.

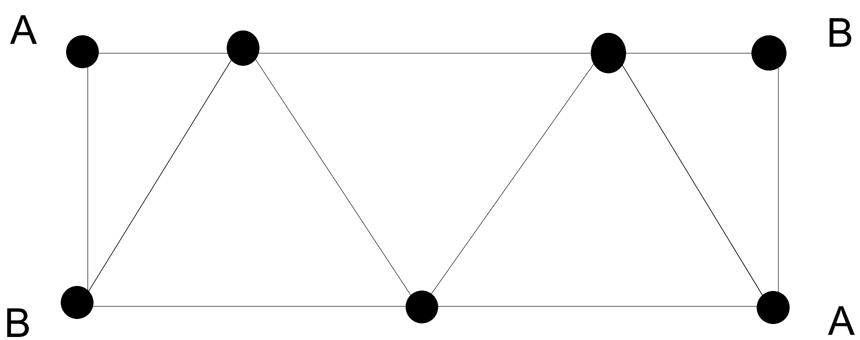
When the external vertices have degree ≥ 6, the octahedron is said to be ***removable***.

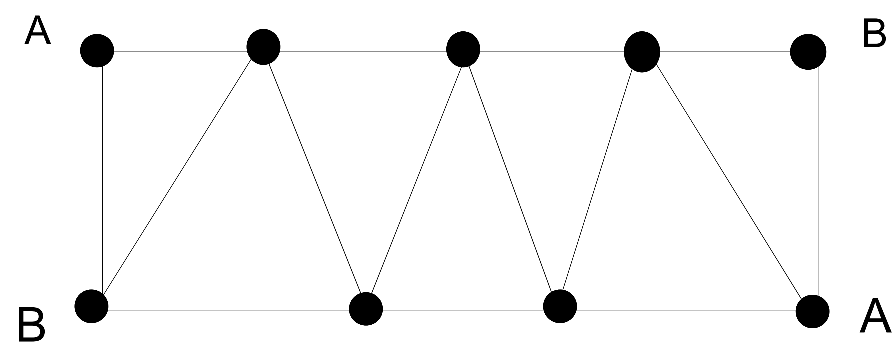
Nakamoto and Negami, 2002

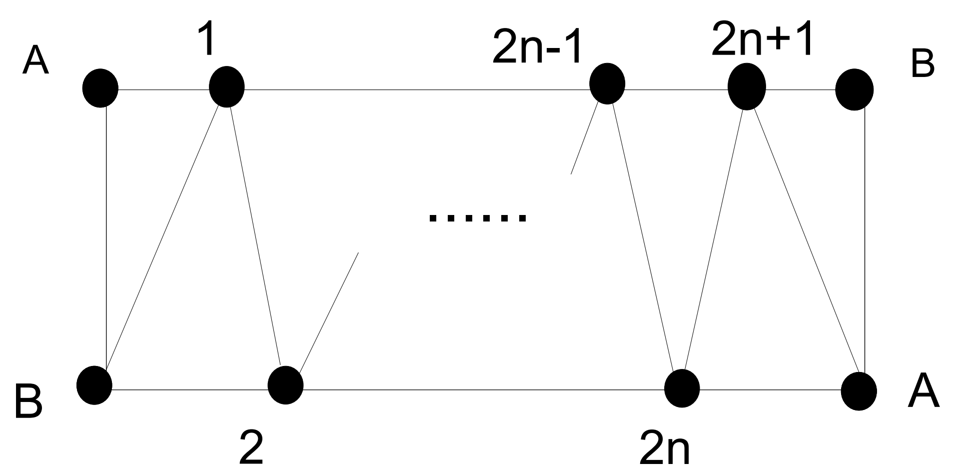
(Θ; 4)-Irreducible triangulations of the Möbius band

- We look for triangulations with no 4-contractible edges.

- Infinitely many triangulations for the Möbius band with no 4-contractible edges exist.

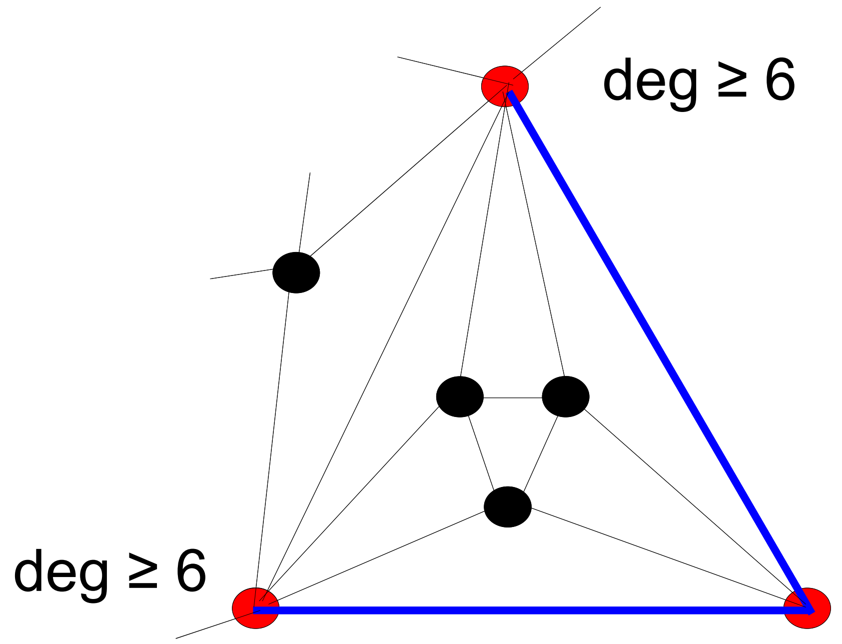


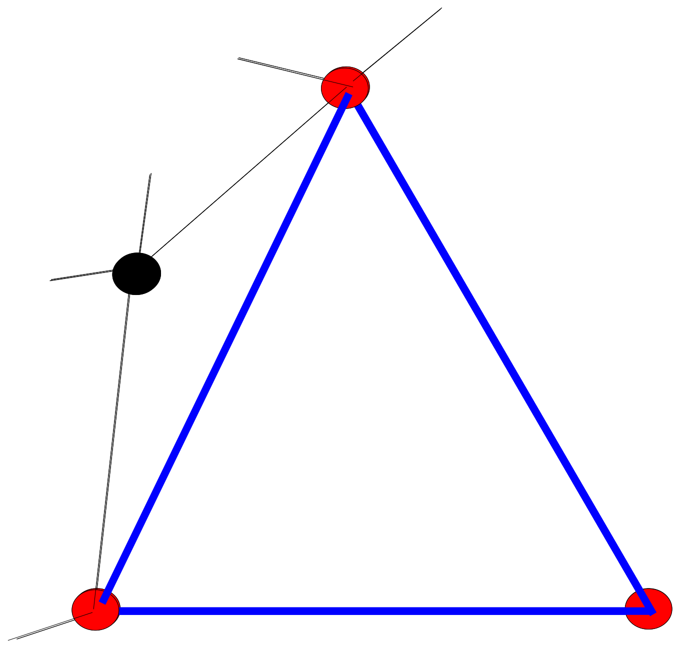




(Θ, 4)-Irreducible triangulations of punctured surfaces

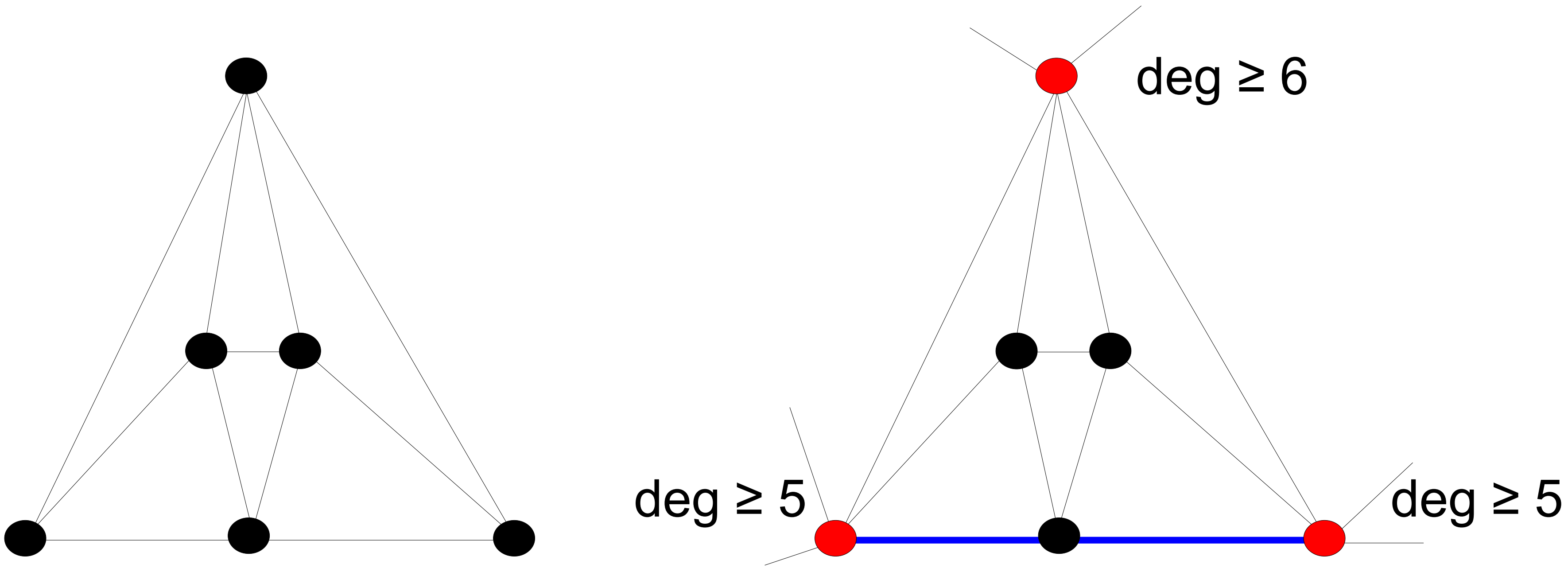
Removing an octahedron in the boundary





(Θ, 4)-Irreducible triangulations of punctured surfaces

Addition of a quasi-octahedron / Removable quasi-octahedron



It is a particular case of a **boundary** face subdivision.

It is equivalent to two consecutive splittings.

The quasi-octahedron is said to be ***removable*** if the degree condition on its external vertices is verified.

(Θ, 4)-Irreducible triangulations of punctured surfaces

(Θ; 4)- reductions

4-contractions of edges,
Removal of octhaedra,
Removal of quasi-octahedra

(Θ; 4)- expansions

4-splitting of vertices,
Addition of octhaedra,
Addition of quasi-octahedra

A triangulation T of a punctured surface F^2 in $F^2(4)$ is said to be ***(Θ; 4)-irreducible*** *if it contains no 4-contractible edge, no removable octahedron nor removable quasi-octahedron.*

Minimum degree ≥ 4

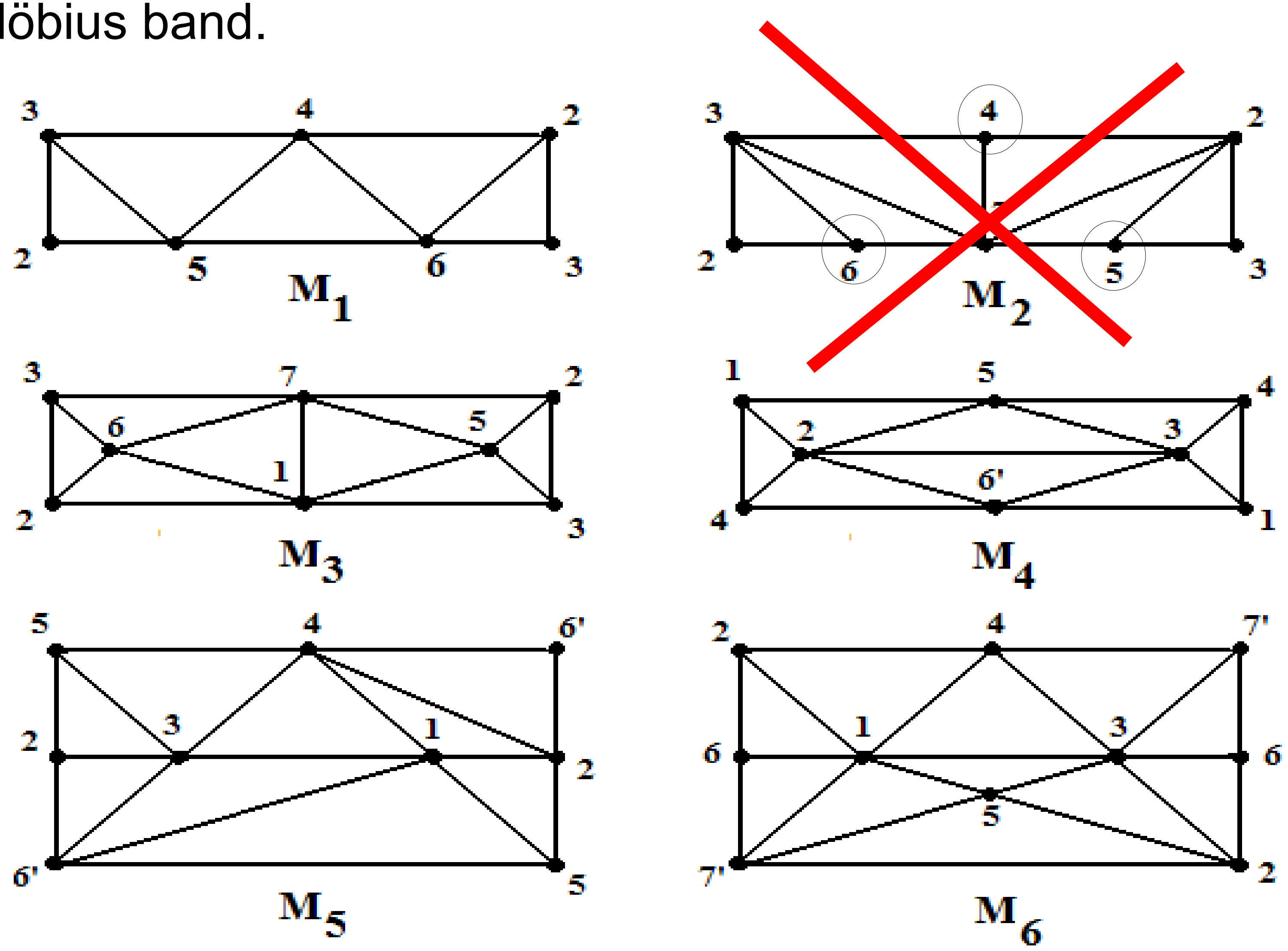
Theorem

Every triangulation T of a ***(punctured) surface*** *F^2 in $F^2(4)$ can be obtained from a (Θ; 4)-irreducible triangulation of F^2 by a sequence of (Θ; 4)-expansions.*

(Θ, 4)-Irreducible triangulations of the Möbius band

Theorem

There exist infinitely many (Θ; 4)-irreducible triangulations of the Möbius band.



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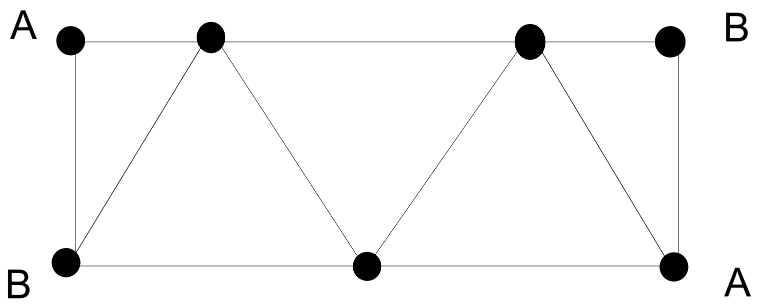

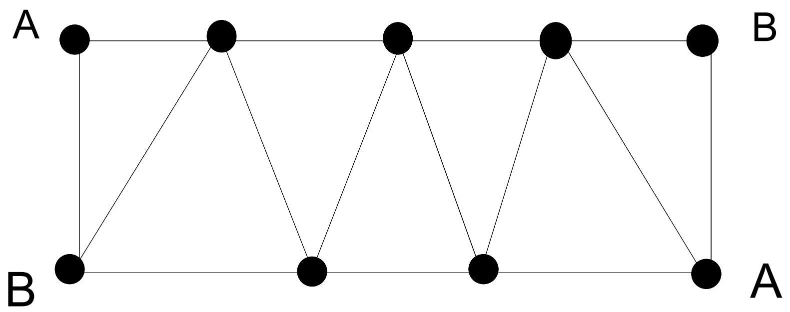

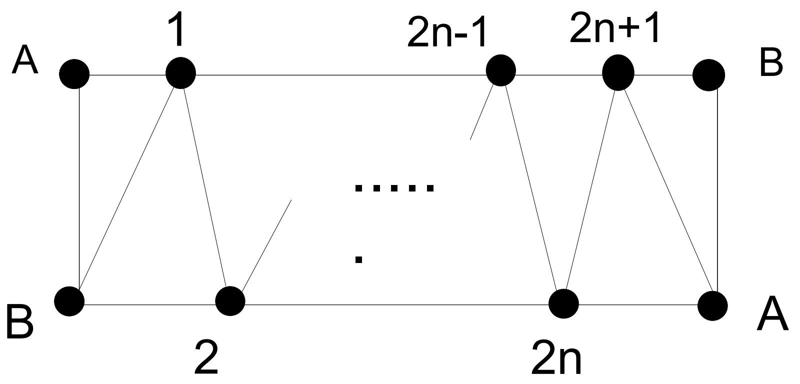

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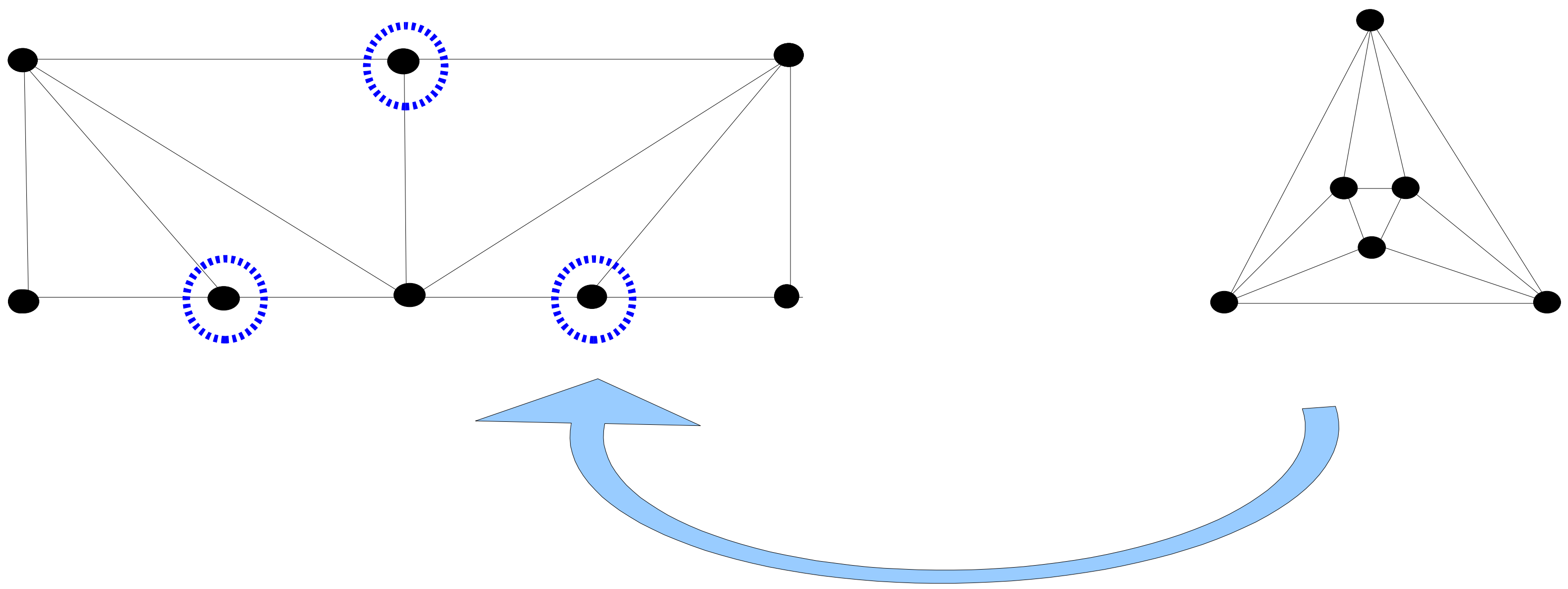

Twuelve new (Θ; 4)-irreducible triangulations of the Möbius band.

(Θ, 4)-Irreducible triangulations of the Möbius band

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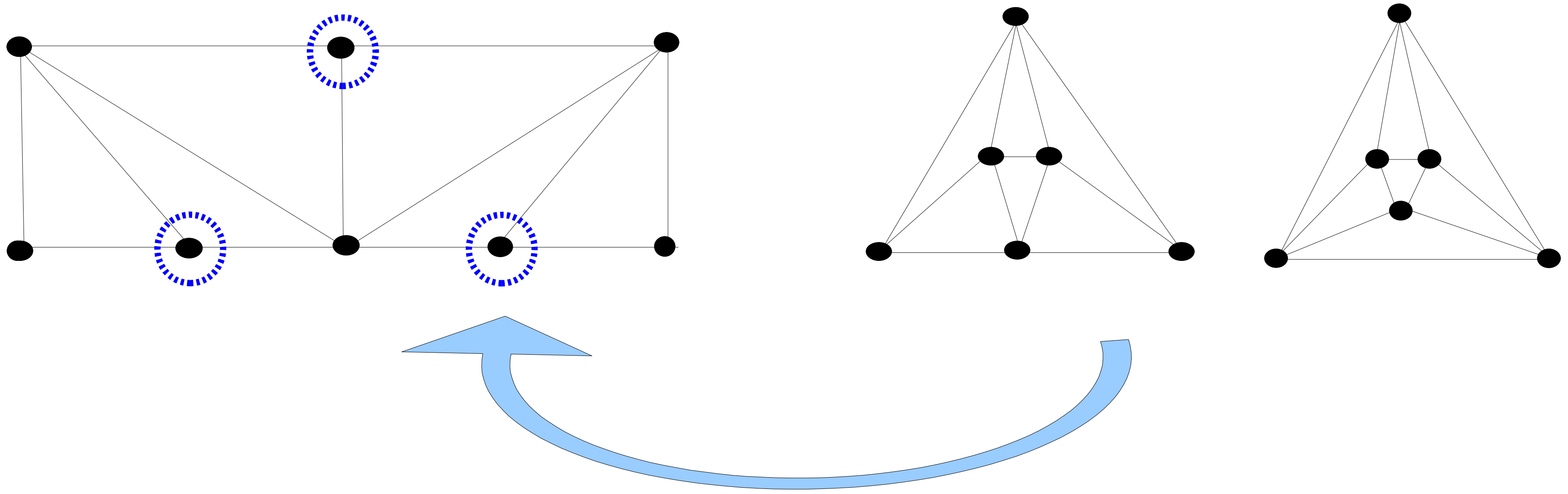

Seven new (Θ; 4)-irreducible triangulations of the Möbius band.

THANK YOU SO MUCH!

¡MUCHAS GRACIAS!