DOMINATION AND ROMAN DOMINATION IN SOME PRODUCT GRAPHS

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Joint work with D. Kuziak and J. A. Rodríguez-Velázquez
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Domination in graphs

- $G = (V, E)$, a simple graph. $S \subseteq V$, set of vertices of $G$.
- $S$ is a dominating set if $N(S) = V$, i.e., every vertex $v \in \bar{S}$ is adjacent to a vertex of $S$.
- $\gamma(G)$, domination number of $G$: minimum cardinality of any dominating set in $G$.

Domination related parameters
- Domination plus conditions on vertices of the dominating set or its complement: Total domination, connected domination, independent domination, etc.
- Conditions over the style of domination: $k$-domination, distance domination, etc.
- Dominating functions: Roman domination, signed domination, minus domination, etc.
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Roman domination

A map $f : V \rightarrow \{0, 1, 2\}$, Roman dominating function for $G$, if for every $v \in V$ with $f(v) = 0$ there exists $u \in N(v)$ such that $f(u) = 2$.

The weight of $f$ if $f(V) = \sum_{v \in V} f(v)$.

$\gamma_R(G)$, Roman domination number of $G$: minimum weight of any Roman dominating function for $G$.

Every Roman dominating function induces three sets $B_0, B_1, B_2$ such that $B_i = \{v \in V : f(v) = i\}$, $i \in \{0, 1, 2\}$.

$\gamma(G) \leq \gamma_R(G) \leq 2\gamma(G)$

$\gamma(G) = \gamma_R(G)$ if and only if $G = K_n$.

$G$ is called a Roman graph if $\gamma_R(G) = 2\gamma(G)$.

There is an open problem related to characterizing all Roman graphs (Roman trees are characterized).
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Product graphs

Cartesian product graphs, $G \square H$
Product graphs

Cartesian product graphs, $G \Box H$

Direct product graphs, $G \times H$
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Strong product graphs, $G \boxtimes H$
Product graphs

Cartesian product graphs, $G \square H$

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Rooted product graphs, $G \circ H$
Domination versus product graphs

Vizing’s conjecture

- One of the most important problems about domination in graphs:
  \[ \gamma(G □ H) \geq \gamma(G)\gamma(H). \]
- Several Vizing-like results for other domination (also not domination related) parameters.
- \[ \Gamma(G □ H) \geq \Gamma(G)\Gamma(H), \gamma(G \times H) \leq 3\gamma(G)\gamma(H), \gamma(G ⊠ H) \leq \gamma(G)\gamma(H), \text{ etc.} \]
- The best approximation to Vizing’s conjecture:
  \[ 2\gamma(G □ H) \geq \gamma(G)\gamma(H) \] (Clark and Suen).

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- \( \gamma_R(G □ H) \geq \gamma(G)\gamma(H). \)
- There were no more results in this topic.
- So, we did it.
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- $S = \{u_1, \ldots, u_t\}$, a dominating set for $G$, $t = \gamma(G)$.
A general bound

- $V_1$ and $V_2$, the vertex sets of $G$ and $H$, respectively.

- $S = \{u_1, ..., u_t\}$, a dominating set for $G$, $t = \gamma(G)$.

- $\Pi = \{A_1, A_2, ..., A_{\gamma(G)}\}$, a vertex partition of $G$ such that $u_i \in A_i$ and $A_i \subseteq N[u_i]$.  

![Diagram showing Cartesian product graphs with vertex sets $A_1, A_2, ..., A_t$ and dominating set $S = \{u_1, ..., u_t\}$]
A general bound

\{\Pi_1, \Pi_2, \ldots, \Pi_{\gamma(G)}\}, a vertex partition of \(G \square H\), such that \(\Pi_i = A_i \times V_2\) for every \(i \in \{1, \ldots, \gamma(G)\}\)
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- \(f = (B_0, B_1, B_2)\), a \(\gamma_R(G \square H)\)-function
A general bound

\[ \{\Pi_1, \Pi_2, \ldots, \Pi_{\gamma(G)}\}, \text{ a vertex partition of } G \square H, \text{ such that } \]
\[ \Pi_i = A_i \times V_2 \text{ for every } i \in \{1, \ldots, \gamma(G)\} \]

\[ f = (B_0, B_1, B_2), \text{ a } \gamma_R(G \square H)\text{-function} \]

For every \( i \in \{1, \ldots, \gamma(G)\}, \) \( f_i : V_2 \rightarrow \{0, 1, 2\}, \) a function on \( H \) such that \( f_i(\nu) = \max\{f(u, \nu) : u \in A_i\}. \)
A general bound

\[ f_i = (X_0^{(i)}, X_1^{(i)}, X_2^{(i)}) \], not a Roman dominating function for \( H \), there is a vertex \( v \in X_0^{(i)} \), \( N(v) \cap X_2^{(i)} = \emptyset \).
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- $f_i = (X_0^{(i)}, X_1^{(i)}, X_2^{(i)})$, not a Roman dominating function for $H$, there is a vertex $v \in X_0^{(i)}$, $N(v) \cap X_2^{(i)} = \emptyset$.

- For every $u \in A_i$, $(u, v)$ is adjacent to some vertex not in $\Pi_i$.
For every $i \in \{1, \ldots, \gamma(G)\}$, we count the number of vertices of $H$ satisfying that they are not adjacent to any vertex with label two (2).
A general bound

- For every $i \in \{1, ..., \gamma(G)\}$, we count the number of vertices of $H$ satisfying that they are not adjacent to any vertex with label two (2).
- For every vertex $v$ of $H$ we count the $G$-cells satisfying that all their vertices are not adjacent to any vertex with label two (2) in the same “column”.
A general bound

- For every $i \in \{1, \ldots, \gamma(G)\}$, we count the number of vertices of $H$ satisfying that they are not adjacent to any vertex with label two (2).
- For every vertex $v$ of $H$ we count the $G$-cells satisfying that all their vertices are not adjacent to any vertex with label two (2) in the same “column”.
- By doing a double sum we get that

$$\gamma_R(G \Box H) \geq \frac{2}{3} \gamma(G) \gamma_R(H)$$
The general bound

For any graphs $G$ and $H$,

$$\gamma_R(G \square H) \geq \frac{2}{3} \gamma(G) \gamma_R(H).$$
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For any graphs $G$ and $H$,

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If $\gamma_R(H) > \frac{3\gamma(H)}{2}$, then

$$\gamma(G \square H) \geq \frac{\gamma(G) \gamma(H)}{2} + \frac{\gamma(G)}{3}.$$
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Roman domination

- $f_1 = (A_0, A_1, A_2)$, $\gamma_R(G)$-function. $f_2 = (B_0, B_1, B_2)$, $\gamma_R(H)$-function.

Then,

$$\gamma_R(G \boxtimes H) \leq \gamma_R(G) + \gamma_R(H) - 2|A_2||B_2|.$$
Roman domination

$f_1 = (A_0, A_1, A_2), \gamma_R(G)$-function. $f_2 = (B_0, B_1, B_2), \gamma_R(H)$-function. Then,

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Then,

\[
\gamma_R(G \boxtimes H) \leq \gamma_R(G) \gamma_R(H) - 2|A_2||B_2|.
\]

Idea of the proof

\( f \) on \( G \boxtimes H \) defined as

\[
f(u, v) = \begin{cases} 
2, & (u, v) \in (A_1 \times B_2) \cup (A_2 \times B_1) \cup (A_2 \times B_2), \\
1, & (u, v) \in A_1 \times B_1, \\
0, & \text{otherwise}. 
\end{cases}
\]
Roman domination

- \( f_1 = (A_0, A_1, A_2), \ \gamma_R(G)\)-function. \( f_2 = (B_0, B_1, B_2), \ \gamma_R(H)\)-function. Then,

\[
\gamma_R(G \boxtimes H) \leq \gamma_R(G)\gamma_R(H) - 2|A_2||B_2|.
\]

Idea of the proof

- \( f \) on \( G \boxtimes H \) defined as

\[
f(u, v) = \begin{cases} 
2, & (u, v) \in (A_1 \times B_2) \cup (A_2 \times B_1) \cup (A_2 \times B_2), \\
1, & (u, v) \in A_1 \times B_1, \\
0, & \text{otherwise}.
\end{cases}
\]

- \( (A_0 \times B_0) \cup (A_0 \times B_2) \cup (A_2 \times B_0) \) is dominated by \( A_2 \times B_2 \),
Roman domination

- \( f_1 = (A_0, A_1, A_2) \), \( \gamma_R(G) \)-function. \( f_2 = (B_0, B_1, B_2) \), \( \gamma_R(H) \)-function. Then,

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- \( f \) on \( G \boxtimes H \) defined as

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- \((A_0 \times B_0) \cup (A_0 \times B_2) \cup (A_2 \times B_0)\) is dominated by \( A_2 \times B_2 \),
- \( A_1 \times B_0 \) is dominated by \( A_1 \times B_2 \) and
Roman domination

- $f_1 = (A_0, A_1, A_2)$, $\gamma_R(G)$-function. $f_2 = (B_0, B_1, B_2)$, $\gamma_R(H)$-function. Then,

\[ \gamma_R(G \boxtimes H) \leq \gamma_R(G)\gamma_R(H) - 2|A_2||B_2|. \]

Idea of the proof

- $f$ on $G \boxtimes H$ defined as

\[
f(u, v) = \begin{cases} 
2, & (u, v) \in (A_1 \times B_2) \cup (A_2 \times B_1) \cup (A_2 \times B_2), \\
1, & (u, v) \in A_1 \times B_1, \\
0, & \text{otherwise.}
\end{cases}
\]

- $(A_0 \times B_0) \cup (A_0 \times B_2) \cup (A_2 \times B_0)$ is dominated by $A_2 \times B_2$,

- $A_1 \times B_0$ is dominated by $A_1 \times B_2$ and

- $A_0 \times B_1$ is dominated by $A_2 \times B_1$. 
Roman domination

- \( f_1 = (A_0, A_1, A_2) \), \( \gamma_R(G) \)-function. \( f_2 = (B_0, B_1, B_2) \), \( \gamma_R(H) \)-function. Then,

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\gamma_R(G \boxtimes H) \leq \gamma_R(G)\gamma_R(H) - 2|A_2||B_2|.
\]

Idea of the proof

- \( f \) on \( G \boxtimes H \) defined as

\[
f(u, v) = \begin{cases} 
2, & (u, v) \in (A_1 \times B_2) \cup (A_2 \times B_1) \cup (A_2 \times B_2), \\
1, & (u, v) \in A_1 \times B_1, \\
0, & \text{otherwise}.
\end{cases}
\]

- \( (A_0 \times B_0) \cup (A_0 \times B_2) \cup (A_2 \times B_0) \) is dominated by \( A_2 \times B_2 \),
- \( A_1 \times B_0 \) is dominated by \( A_1 \times B_2 \) and
- \( A_0 \times B_1 \) is dominated by \( A_2 \times B_1 \).
- \( f \) is a Roman dominating function on \( G \boxtimes H \).
Index

1. Introduction
2. Cartesian product graphs
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Domination

- $G$, graph of order $n \geq 2$. $H$, graph with root $v$ and at least two vertices. If $v$ does not belong to any $\gamma(H)$-set or $v$ belongs to every $\gamma(H)$-set, then

$$\gamma(G \circ H) = n\gamma(H).$$
**Domination**

- $G$, graph of order $n \geq 2$. $H$, graph with root $v$ and at least two vertices. If $v$ does not belong to any $\gamma(H)$-set or $v$ belongs to every $\gamma(H)$-set, then
  \[ \gamma(G \circ H) = n\gamma(H). \]

- $G$, graph of order $n \geq 2$. Then for any graph $H$ with root $v$ and at least two vertices,
  \[ \gamma(G \circ H) \in \{n\gamma(H), n(\gamma(H) - 1) + \gamma(G)\}. \]
Roman domination

\[ G, \text{ graph of order } n \geq 2. \text{ Then for any graph } H \text{ with root } v \text{ and at least two vertices,} \]

\[ n(\gamma_R(H) - 1) + \gamma(G) \leq \gamma_R(G \circ H) \leq n\gamma_R(H). \]
Roman domination

- $G$, graph of order $n \geq 2$. Then for any graph $H$ with root $v$ and at least two vertices,

$$n(\gamma_R(H) - 1) + \gamma(G) \leq \gamma_R(G \circ H) \leq n\gamma_R(H).$$

Tightness of the bounds

- If for every $\gamma_R(H)$-function $f = (B_0, B_1, B_2)$ is satisfied that $f(v) = 0$, then

$$\gamma_R(G \circ H) = n\gamma_R(H).$$
Roman domination

- **G**, graph of order $n \geq 2$. Then for any graph $H$ with root $v$ and at least two vertices,

$$n(\gamma_R(H) - 1) + \gamma(G) \leq \gamma_R(G \circ H) \leq n\gamma_R(H).$$

Tightness of the bounds

- If for every $\gamma_R(H)$-function $f = (B_0, B_1, B_2)$ is satisfied that $f(v) = 0$, then

$$\gamma_R(G \circ H) = n\gamma_R(H).$$

- If there exist two $\gamma_R(H)$-functions $h = (B_0, B_1, B_2)$ and $h' = (B'_0, B'_1, B'_2)$ such that $h(v) = 1$ and $h'(v) = 2$, then

$$\gamma_R(G \circ H) = n(\gamma_R(H) - 1) + \gamma(G).$$
- $G$, graph of order $n \geq 2$ and $H$, graph with root $v$ and at least two vertices.
Rooted product graphs

- $G$, graph of order $n \geq 2$ and $H$, graph with root $v$ and at least two vertices.
- If for every $\gamma_R(H)$-function $f$ is satisfied that $f(v) = 1$, then

$$
\gamma_R(G \circ H) = n(\gamma_R(H) - 1) + \gamma_R(G).
$$
THANKS!!!