How false is the Hirsch Conjecture?

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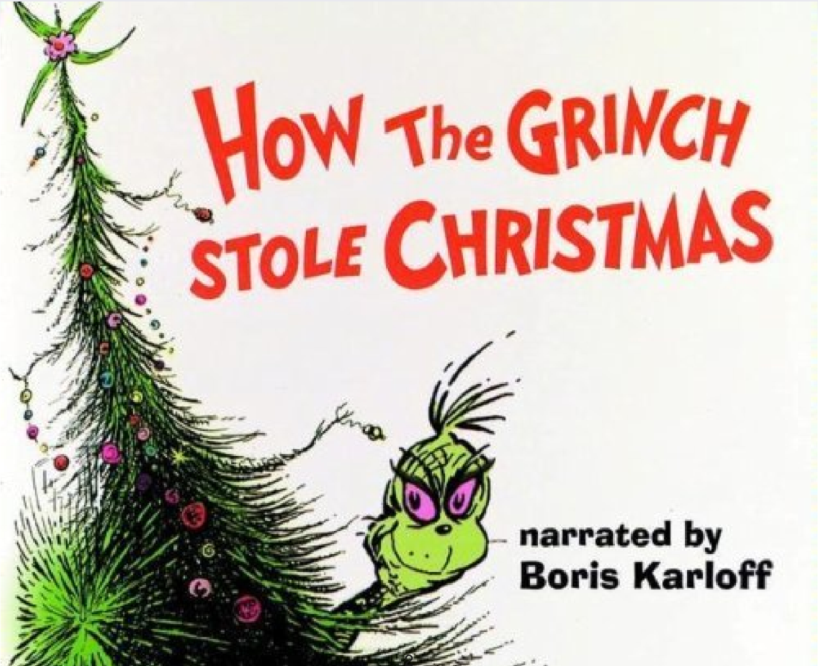
HOW The GRINCH
STOLE CHRISTMAS
narrated by
Boris Karloff

Polyhedra and polytopes

Definition

A (convex) polyhedron P is the intersection of a finite family of affine half-spaces in $\mathbb{R}^d$.

The dimension of P is the dimension of its affine hull.

Polyhedra and polytopes

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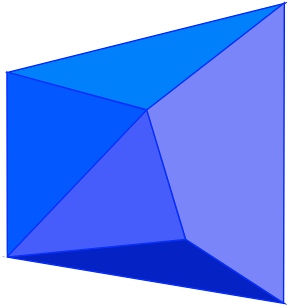
A (convex) polytope P is the convex hull of a finite set of points in $\mathbb{R}^d$.



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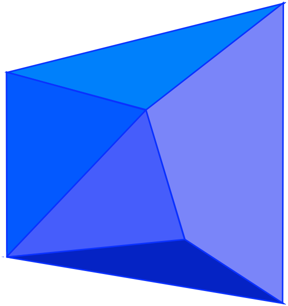
Polyhedra and polytopes

Polytope = bounded polyhedron.



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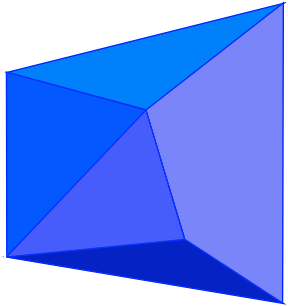

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Faces of P

Let P be a polytope (or polyhedron) and let

$$H = \{x \in \mathbb{R}^d : a_1 x_1 + \cdots a_d x_d \leq a_0\}$$

be an affine half-space.

If $P \subset H$ we say that $\partial H \cap P$ is a face of P.

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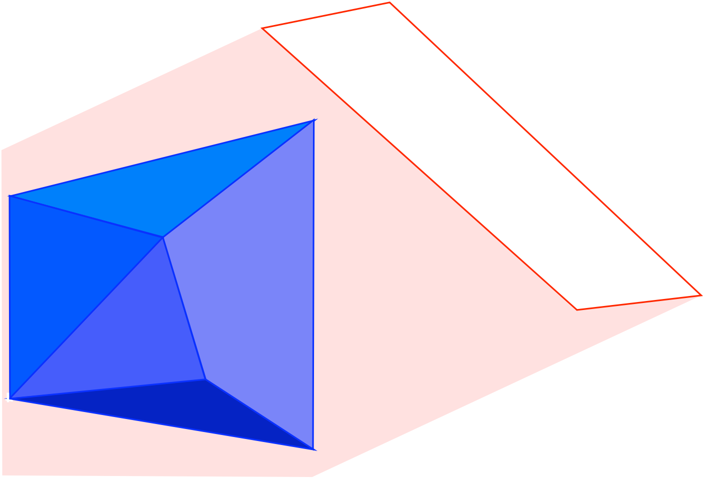
Faces of P

The “empty face” of P.



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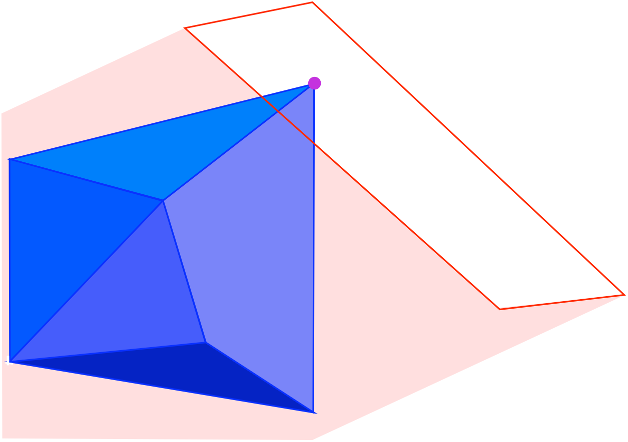
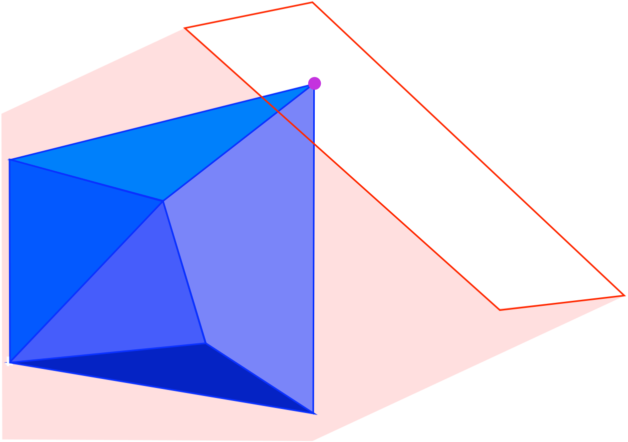


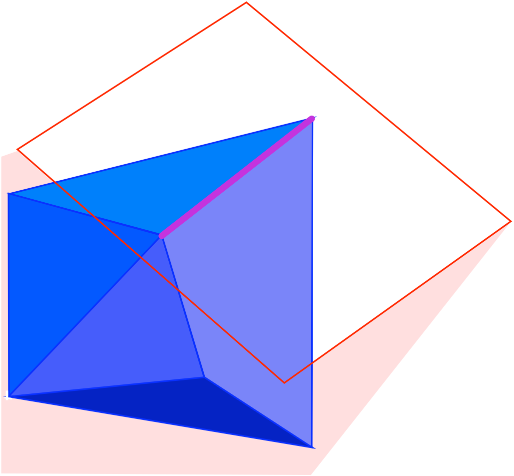
Faces of P

Faces of dimension 0 are called vertices.



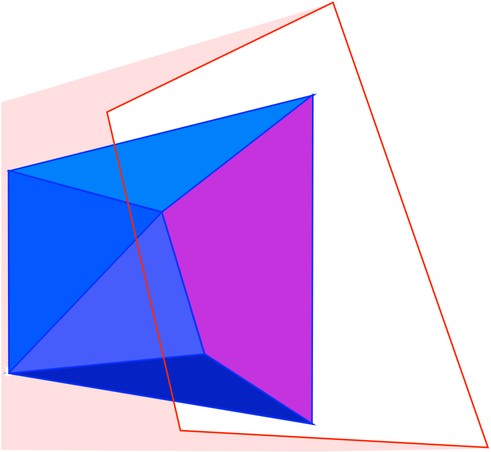
Faces of P

Faces of dimension 1 are called edges.



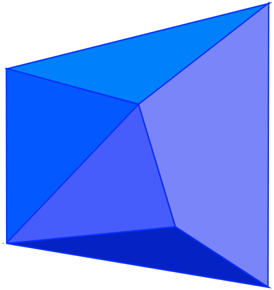
Faces of P

Faces of dimension $d-1$ (codimension 1) are called facets.



The graph of a polytope

Vertices and edges of a polytope P form a (finite, undirected) graph.



The distance $d(u, v)$ between vertices u and v is the length (number of edges) of the shortest path from u to v.

For example, $d(u, v) = 2$.

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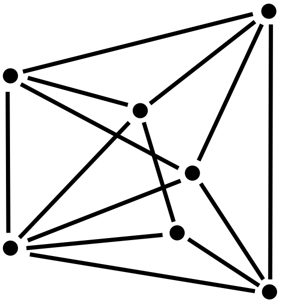

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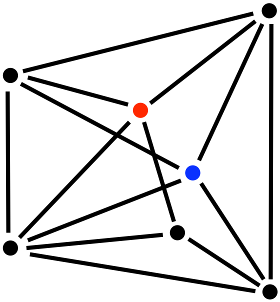

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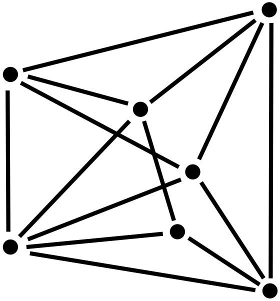

The diameter of $G(P)$ (or of P) is the maximum distance among its vertices:

$$\delta(P) := \max\{d(u, v) : u, v \in V\}.$$

The Hirsch conjecture

Let $\delta(P)$ denote the diameter of the graph of a polytope P.

Conjecture: Warren M. Hirsch (1957)

For every polyhedron P with n facets and dimension d,

$$\delta(P) \leq n - d.$$

polytope	faces	dimension	$n - d$	diameter
cube	6	3	3	3
dodecahedron	12	3	9	5
octahedron	8	3	5	2
k-prism	$k+2$	3	$k-1$	$\lfloor k/2 \rfloor + 1$
n-cube	$2n$	n	n	n

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Fifty three years later...

Theorem (S. 2010+)

There is a 43-dim. polytope with 86 facets and diameter ≥ 44.

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Theorem (Matschke-S.-Weibel 2011+)

There is a 20-dim. polytope with 40 facets and diameter ≥ 21.

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Motivation: linear programming

- The set of feasible solutions $P = \{x \in \mathbb{R}^d : Mx \leq b\}$ is a polyhedron P with (at most) n facets and d dimensions.
- The optimal solution (if it exists) is always attained at a vertex.
- The simplex method [Dantzig 1947] solves linear programming by starting at any feasible vertex and moving along the graph of P, in a monotone fashion, until the optimum is attained.
- In particular, the Hirsch conjecture is related to the question of whether the simplex method is a polynomial time algorithm (for some pivot rule).

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> *The simplex method has remained, if not the method of choice, a method of choice, usually competitive with, and on some classes of problems superior to, the more modern approaches.*
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What do we know?

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Theorem [Kalai-Kleitman 1992]

$$H(n,d) \leq n^{\log_2 d + 2}, \qquad \forall n, d.$$

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The d-step Theorem

Theorem (Klee-Walkup, 1967)

Let P be a polytope of dimension d, with n facets and diameter δ. Then there is another polytope P' of dimension $d+1$, with $n+1$ facets and diameter $\geq \delta$.

Corollary (d-step theorem)

For each $n > d \in \mathbb{N}$, let $H(n,d)$ denote the maximum diameter among d-polytopes with n facets. Then

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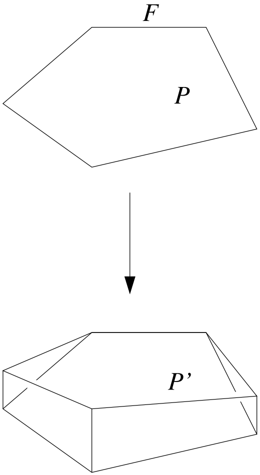

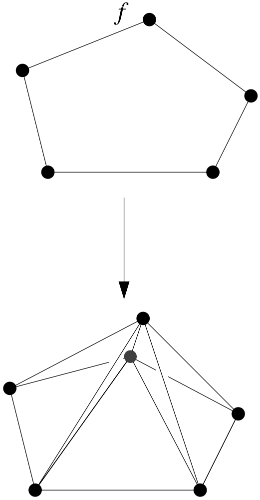

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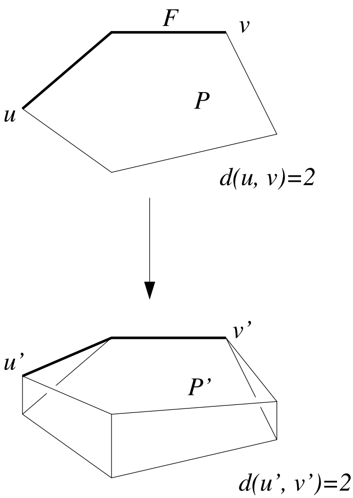

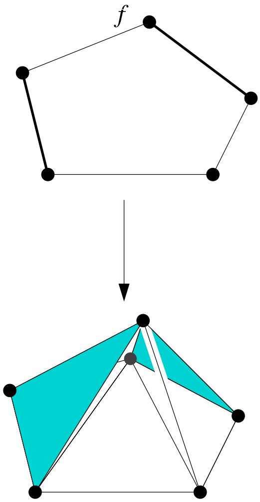

The construction

The construction of counter-examples has two ingredients:

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2. The construction of a prismatoid of dimension 5 and "width" 6.

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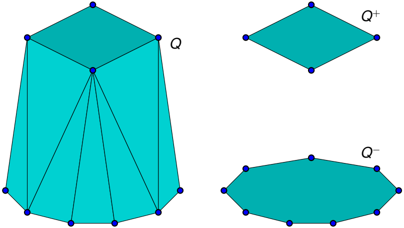
A *prismatoid* is a polytope Q with two (parallel) facets Q^+ and Q^- containing all vertices.



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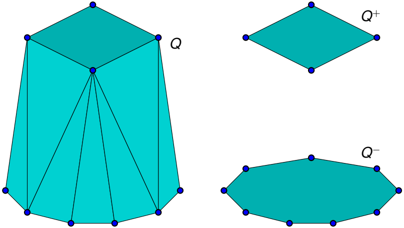

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Theorem (Strong d-step theorem, prismatoid version)

Let Q be a prismatoid of dimension d, with $n > 2d$ vertices and width δ. Then there is another prismatoid Q' of dimension $d+1$, with $n+1$ vertices and width $\delta+1$.

That is: we can increase the dimension, width and number of vertices of a prismatoid, all by one, until $n = 2d$.

Corollary

In particular, if a prismatoid Q has width $> d$ then there is another prismatoid Q' (of dimension $n-d$, with $2n-2d$ vertices, and width $\geq \delta + n - 2d > n-d$) that violates (the dual of) the Hirsch conjecture.

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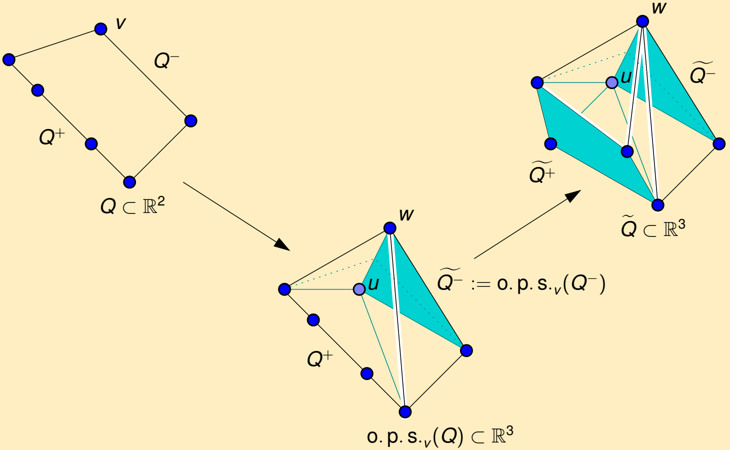
The strong d-step Theorem

Proof.



□

Width of prismatoids

So, to disprove the Hirsch Conjecture we only need to find a prismatoid of dimension d and width larger than d. *Its number of vertices and facets is irrelevant...*

Question

Do they exist?

- 3-prismatoids have width at most 3 (exercise).
- 4-prismatoids have width at most 4 [S.-Stephen-Thomas, 2011].
- 5-prismatoids of width 6 exist [S., 2010] with 25 vertices [Matschke-S.-Weibel 2011+].
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So, to disprove the Hirsch Conjecture we only need to find a prismatoid of dimension d and width larger than d. *Its number of vertices and facets is irrelevant...*

Question

Do they exist?

- 3-prismatoids have width at most 3 (exercise).
- 4-prismatoids have width at most 4 [S.-Stephen-Thomas, 2011].
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A 5-prismatoid of width > 5

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The following prismatoid Q, of dimension 5 and with 48 vertices, has width six.

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$$Q := \operatorname{conv}\left\{ \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} \pm 18 & 0 & 0 & 0 & 1 \\ 0 & \pm 18 & 0 & 0 & 1 \\ 0 & 0 & \pm 45 & 0 & 1 \\ 0 & 0 & 0 & \pm 45 & 1 \\ \pm 15 & \pm 15 & 0 & 0 & 1 \\ 0 & 0 & \pm 30 & \pm 30 & 1 \\ 0 & \pm 10 & \pm 40 & 0 & 1 \\ \pm 10 & 0 & 0 & \pm 40 & 1 \end{pmatrix} \end{array} \quad \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 & x_4 & x_5 \end{matrix} \\ \begin{pmatrix} 0 & 0 & 0 & \pm 18 & -1 \\ 0 & 0 & \pm 18 & 0 & -1 \\ \pm 45 & 0 & 0 & 0 & -1 \\ 0 & \pm 45 & 0 & 0 & -1 \\ 0 & 0 & \pm 15 & \pm 15 & -1 \\ \pm 30 & \pm 30 & 0 & 0 & -1 \\ \pm 40 & 0 & \pm 10 & 0 & -1 \\ 0 & \pm 40 & 0 & \pm 10 & -1 \end{pmatrix} \end{array} \right\}$$

A 5-prismatoid of width > 5

Theorem (S. 2010)

The following prismatoid Q, of dimension 5 and with 48 vertices, has width six.

Corollary

There is a 43-dimensional polytope with 86 facets and diameter (at least) 44.

Smaller 5-prismatoids of width > 5

With the same ideas

Theorem (Matschke-Santos-Weibel, 2011)

There is a 5-prismatoid with 25 vertices and of width 6.

Corollary

There is a non-Hirsch polytope of dimension 20 with 40 facets.

This one has been explicitly computed. It has $36,442$ vertices, and diameter 21.

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poly20dim21.ext

```
V-representation
begin
 40 21 rational
1 1 0 0 21 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0 0
1 1 0 0 16 -8 0 0 0 0 0 0 0 0 0 0 0 1 1 0 0
1 1 0 0 0 -25 0 0 0 0 0 0 0 0 0 0 0 0 1 1 0
1 1 0 0 -16 -10 0 0 0 0 0 0 0 0 0 0 0 0 0 1 1
1 1 0 0 -21 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 1
1 1 0 0 -20 3 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 0 0 0 39 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 0 0 19 7 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 3/100 -1/50 0 -23 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 -3/100 -1/50 0 37 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 1 -3/2000 7/2000 0 388/10 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 10000000000 100000000000 100000000000 100000000000 100000000000 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 -10000000 0 0 0 0 0 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 -10000000 0 0 0 0 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 -10000000 0 0 0 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 -10000000000 0 0 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 10000000000 -100000000000 0 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 10000000000 100000000000 -100000000000 0 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 10000000000 100000000000 100000000000 -100000000000 0 1 0 0 0 0 0 0
1 1 3/2000 7/2000 0 -248/10 10000000 10000000 10000000 10000000000 100000000000 100000000000 100000000000 -100000000000 1 0 0 0 0 0 0
1 -1 30 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0 0
1 -1 4 -15 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0 0
1 -1 0 -33/2 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0 0
1 -1 -1 -16 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0 0
1 -1 -55/2 0 0 0 0 0 0 0 0 0 1 1 0 0 0 0 0 0 0
1 -1 -17 18 0 0 0 0 0 0 0 0 0 1 0 0 0 0 0 0 0
1 -1 0 38 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 22 17 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 -10 0 1/5 -1/5 0 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 2999/100 0 -3/25 -1/5 0 0 1 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 299999/10000 0 0 1/100 0 1 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 -2745/100 0 1/5000 1/800 1 0 0 0 0 0 0 0 0 0 0 0 0 0 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 10000000 10000000 100000000 100000000 1000000000
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 -100000 0 0 0 0 0 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 -10000000 0 0 0 0 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 -10000000 0 0 0 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 10000000 -10000000 0 0 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 10000000 10000000 -100000000 0 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 10000000 10000000 100000000 -100000000 0
1 -1 -27 0 1/500 -1/80 0 0 0 0 0 0 0 0 100000 10000000 10000000 10000000 100000000 100000000 -1000000000
end
allbases
printcobasis
```

Many non-Hirsch polytopes

Once we have a non-Hirsch polytope we can derive more via:

1. Products of several copies of it (dimension increases).
2. Gluing several copies of it (dimension is fixed).

To analyze the asymptotics of these operations, we call excess of a d-polytope P with n facets and diameter δ the number

$$\epsilon(P) := \frac{\delta}{n-d} - 1 = \frac{\delta - (n-d)}{n-d}.$$

E. g.: The excess of our non-Hirsch polytope with $n - d = 20$ and with diameter 21 is

$$\frac{21-20}{20} = 5\%.$$

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1. Taking products preserves the excess: for each $k \in \mathbb{N}$, there is a non-Hirsch polytope of dimension $20k$ with $40k$ facets and with excess equal to $0.05 = 5\%$.
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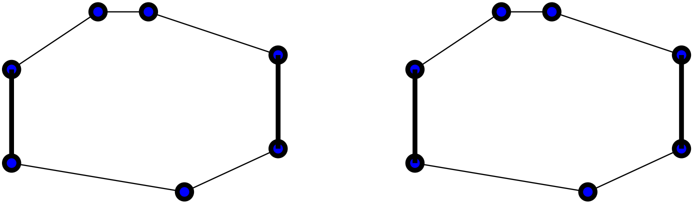

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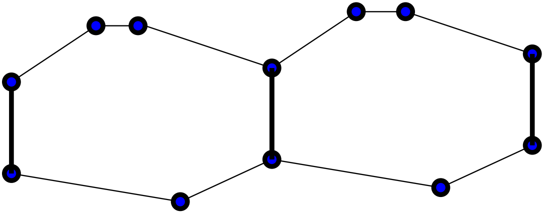

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$$\delta = \delta_1 + \delta_2 - 1$$

$$\frac{\delta_1}{n_1 - d} - 1 = \frac{\delta_2}{n_2 - d} - 1 = \epsilon \qquad \Rightarrow \qquad \frac{\delta}{n-d} - 1 = \epsilon - \frac{1}{(n_1 - d) + (n_2 - d)}.$$

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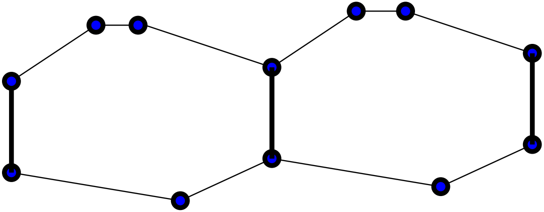

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Corollary

For each $k \in \mathbb{N}$ there is an infinite family of non-Hirsch polytopes of fixed dimension $20k$ and with excess (tending to)

$$0.05\left(1 - \frac{1}{k}\right).$$

The excess of a prismatoid

But we know there are “worst” prismatoids: 5-prismatoids of arbitrarily large width. Will those produce non-Hirsch polytopes with worst excess?

To analyze the asymptotics of this, let us call *excess* of a prismatoid of width δ with n vertices and dimension d the quantity

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Lemma

Via the strong d-step Theorem, a prismatoid of a certain excess produces non-Hirsch polytopes of that same excess.

Proof.

The dimension, number of facets and diameter of the non-Hirsch polytope produced by the strong d-step Theorem are

$$n-d, \qquad 2(n-d), \qquad \delta+(n-2d).$$

So, its excess is

$$\frac{\delta+(n-2d)-(n-d)}{n-d}=\frac{\delta-d}{n-d}.$$

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In dimension 5, we know how to construct polytopes of arbitrarily large width $\delta \sim \sqrt(n)$... but their excess tends to zero:

$$\lim \frac{\delta - 5}{n - 5} = \lim \frac{\sqrt{n} - 5}{n - 5} = 0.$$

Let us be optimistic and suppose that we could construct 5-prismatoids with n vertices and linear width $\simeq \alpha n$.

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In fixed dimension, certainly not:

Theorem

The width of a d-dimensional prismatoid with n vertices cannot exceed $2^{d-3}n$.

Proof.

This is a general result for the (dual) diameter of a polytope [Barnette, Larman, ∼1970]. □

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In fact, in dimension five we can tighten the upper bound a little bit:

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Corollary

Using the Strong d-step Theorem for 5-prismatoids it is impossible to violate the Hirsch conjecture by more than 33%.

If you cannot prove it, generalize it. . .

More general setting

Instead of looking at (simplicial) polytopes, why not look at the maximum diameter of more general complexes?

- Strongly connected pure simplicial complexes. $H_C(n, d)$
- Pseudo-manifolds (w. or wo. bdry). $H_{\overline{pm}}(n, d)$, $H_{pm}(n, d)$
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Remark, in all definitions of $H_\bullet(n, d)$, n is the number of vertices and $d - 1$ is the dimension.

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Some easy remarks and a toy example

There are the following relations:

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In dimension two:

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1. Without loss of generality assume $n = 3k + 1$.
2. With the first $2k + 1$ vertices, construct k disjoint cycles of length $2k + 1$ (That is, decompose K_{2k+1} into k disjoint Hamiltonian cycles).
3. Remove an edge from each cycle to make it a chain, and join each chain to each of the remaining k vertices.
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2. Take the join Δ^{*k} of k copies of Δ. Δ^{*k} is a complex of dimension $kd - 1$, with kn vertices and whose dual graph is a k-dimensional grid of size $H_C(n, d)$. (It has $(H_C(n, d) + 1)^k$ maximal simplices).
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A special class of complexes

Definition

A connected layer family (CLF) of rank d on n symbols is a pure simplicial complex Δ of dimension $d-1$ with n vertices, together with a map

$$\lambda : \text{facets}(\Delta) \to \mathbb{Z}$$

with the following property: for every simplex (of whatever dimension) $\tau \in \Delta$ the values taken by λ in the star of τ form an interval.

The length of a CLF is the difference between the maximum and the minimum values taken by λ.

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More generally, $H_{clf}(n, d)$ is an upper bound for the diameter of all complexes with *connected links*.

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1. $H_{clf}(n,d) \geq H_{\overline{M}}(n,d) \geq H(n,d)$.
2. $H_{clf}(n,d) \leq n^{\log_2 d + 2}$. *(Kalai-Kleitman bound)*
3. $H_{clf}(n,d) \leq 2^{d-2} n$. *(Barnette-Larman bound)*
4. $H_{clf}(n, n/4) \geq \Omega(n^2 / \log n)$.

This implies, for example:

Corollary (of part 3)

A surface (with or without boundary) cannot have diameter greater than $2n$.

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Do surfaces satisfy the Hirsch conjecture? (Those without boundary do).

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$H_{clf}(n,d) \le n^{\log_2 d+2}$ (Kalai-Kleitman bound)

The Kalai-Kleitman bound follows from the following recursion:

$$H_{clf}(n,d) \le H_{clf}(\lfloor n/2 \rfloor, d) + H_{clf}(n-1, d-1) + 2.$$

To prove the recursion:

- Let u and v be two simplices. For each $i \in \mathbb{N}$, let U_i be the i-neighborhood of u (the subcomplex consisting of all layers at distance at most i from u). Call V_j the j-neighborhood of v.
- Let i_0 and j_0 be the smallest values such that U_{i_0} and V_{j_0} contain more than half of the vertices. This implies $i_0 - 1$ and $j_0 - 1$ are at most $H_{clf}(\lfloor n/2 \rfloor, d)$.
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- Let $u' \in U_{i_0}$ and $v' \in V_{j_0}$ having a common vertex. Then:

$$d(u', v') \le H_{clf}(n-1, d-1).$$

So: $d(u,v) \le d(u,u') + d(u',v') + d(u,v) \le$
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$H_{clf}(n,d) \leq n^{\log_2 d+2}$ (Kalai-Kleitman bound)

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Definition

A connected layer multifamily (CLMF) of rank d on n symbols is the same as a CLF, except we allow a pure simplicial multicomplex Δ (simplices are multisets of vertices, with repetitions allowed)

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λ	3	4	5	6	7	8	9	10	11	12
Δ	111	112	113	114	124	134	144	244	344	444
			122	123	133	224	234	334		
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Hähnle's Conjecture

"Complete" and "injective" clmf are two extremal cases. It turns out that in these two cases:

Theorem (Hähnle et al@polymath3, 2010)

A Connected Layer (Multi)-Family with λ injective or Δ complete cannot have length greater than $d(n-1)$.

This suggests the following conjecture

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TO BE CONTINUED???

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