Polytopes of combinatorial degree 1

Arnau Padrol

Universitat Politècnica de Catalunya

11 July 2012

VIII Jornadas de Matemática Discreta y Algorítmica

joint work with Benjamin Nill

Polytopes and point configurations

What is a polytope?

Polytope Convex hull of a finite point set $A \subset \mathbb{R}^d$

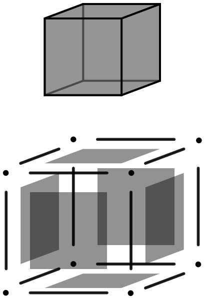
$\Leftrightarrow$

Bounded intersection of finitely many half-spaces

Face Intersection with a supporting hyperplane

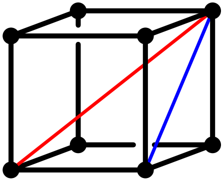
Vertices, Edges, Facets Faces of dimension 0, 1, $d-1$.

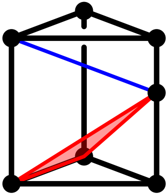
$$F \text{ face of } \operatorname{conv}\{A\} \Rightarrow F = \operatorname{conv}\{A \cap F\}$$



Interior faces

Given a point configuration A, $S \subseteq A$ is an *interior face* of a A if $\operatorname{conv}(S)$ does not lie on the boundary of $\operatorname{conv} A$.





Definition

The *combinatorial degree* of a point configuration is

$$\deg_c(A) = d + 1 - k,$$

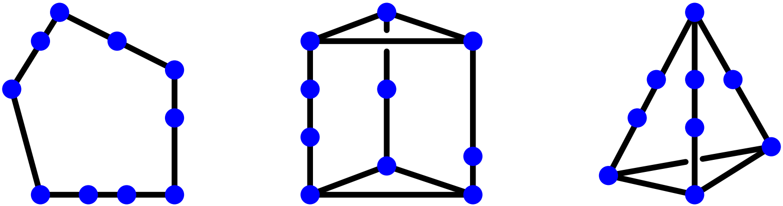
where k is the smallest cardinality of an interior face of A.

Point configurations of degree 1

Theorem (Nill & P.)

$A \subset \mathbb{R}^d$ has $\deg_c(A) \leq 1$ if and only if A is a k-fold pyramid over:

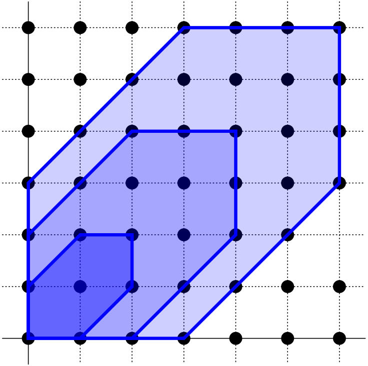
1. *a polygon with points on its boundary,*
2. *a prism over a simplex with points on the "vertical" edges,*
3. *a simplex with a vertex v and points on its adjacent edges.*



Lattice polytopes

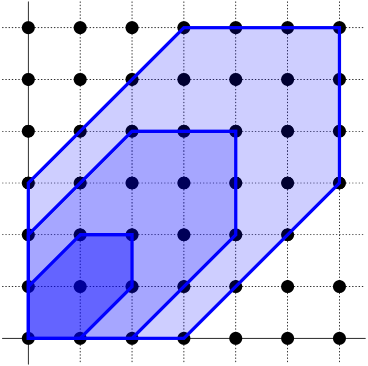
Lattice points in lattice polytopes

$|kP \cap \mathbb{Z}^d|$: # lattice points in multiples of a lattice d-polytope P:



Lattice points in lattice polytopes

$|kP \cap \mathbb{Z}^d|$: # lattice points in multiples of a lattice d-polytope P:



Ehrhart polynomial & series

Ehrhart:

$$k \mapsto |kP \cap \mathbb{Z}^d|$$

is a polynomial.

Moreover,

$$\sum_{k \geq 0} |(kP) \cap \mathbb{Z}^d| \, t^k = \frac{h_P^*(t)}{(1-t)^{d+1}},$$

where h_P^* is a polynomial of degree $\leq d$.

Ehrhart theory: the h^*-polynomial

Definition

The *degree* of P is $\deg(P) = \deg(h^*_P)$.

Proposition (Batyrev & Nill)

$d + 1 - \deg(P)$ is min $k \in \mathbb{Z}_{>0}$ *such that kP has interior lattice points.*

$P \cap \mathbb{Z}^d$ cannot have interior faces of cardinality $< d + 1 - \deg(P)$!

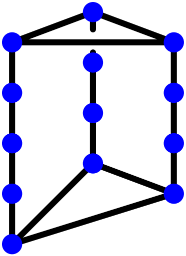
$$\deg_c(P \cap \mathbb{Z}^d) \leq \deg(P)$$

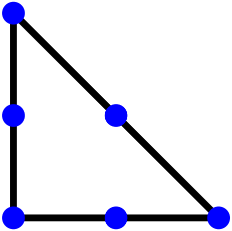
Lattice polytopes of degree 1

Theorem (Batirev & Nill 2007)

Let P be lattice polytope. Then $\deg(P) \leq 1$ if and only if P is

1. *A Lawrence prism or,*
2. *an exceptional simplex.*





Triangulations

The generalized lower bound theorem

Definition

P simplicial d-polytope

f-vector $f_i(P)$: # of i-faces of P.

h-vector $\sum_{0\leq i\leq d} h_i(P)x^{d-i} = \sum_{0\leq i\leq d} f_{i-1}(P)(x-1)^{d-i}$.

Generalized Lower Bound ~~Conjecture~~ *Theorem* **[McMullen&Walkup 1971, Stanley 1980, Murai&Nevo 2012]**

Let P be a simplicial d-polytope, then

1. $h_i \geq h_{i-1}$ for all $2 \leq i \leq \lfloor d/2 \rfloor$,
2. $h_{i+1} = h_i$ if and only if P can be triangulated without interior faces of cardinality $\leq d - i$.

All triangulations of P avoid all interior faces of cardinality $d - \deg_c P$.

Lower bound theorem for balls

Theorem (Lower bound theorem for balls)

The size of a simplicial d-ball $\mathcal{B}$ with n vertices is $|\mathcal{B}| \geq n - d$.

$|\mathcal{B}| = n - d \Leftrightarrow \mathcal{B}$ *has no interior $(d-2)$-cell.*

Corollary

$\deg_c(A) \leq 1$ *if and only if all triangulations of A are minimal.*

Tverberg's Theorem

The m-core and m-split

Definition

A set of n points in $\mathbb{R}^r$, $x \in \mathbb{R}^r$

- x has *depth m* if $\forall$ closed halfspace $\bar{h}$: $x \in \bar{h} \Rightarrow |\bar{h} \cap A| \geq m$.
- x is *m-divisible* if there are m disjoint subsets of A $S_1, \ldots, S_m$ with $x \in \operatorname{conv} S_i$.
- $\mathcal{C}_m(A)$: depth m points.
- $\mathcal{D}_m(A)$: m-divisible points.

Theorem (Tverberg's Theorem)

$$\mathcal{D}_m(A) \neq \emptyset \text{ if } n \geq (m-1)(d+1)+1.$$

$$\mathcal{D}_m(A) \subsetneq \mathcal{C}_m(A)$$

Reformulation

A consequence of our theorem:

Theorem

In $\mathbb{R}^r$*, for* $|A| = n$*,*

$$\mathcal{C}_{n-r-1} \subseteq \mathcal{D}_{n-r-2}$$

Reformulation

A consequence of our theorem:

Theorem

In $\mathbb{R}^r$*, for* $|A| = n$,

$$\mathcal{C}_{n-r-1} \subseteq \mathcal{D}_{n-r-2}$$

Conjecture

In $\mathbb{R}^r$*, for* $|A| = n$,

$$\mathcal{C}_{n-r-\delta} \subseteq \mathcal{D}_{n-r-2\delta}$$

Reformulation

A consequence of our theorem:

Theorem

In $\mathbb{R}^r$, *for* $|A| = n$,

$$\mathcal{C}_{n-r-1} \subseteq \mathcal{D}_{n-r-2}$$

Conjecture

In $\mathbb{R}^r$, *for* $|A| = n$,

$$\mathcal{C}_{n-r-\delta} \subseteq \mathcal{D}_{n-r-2\delta}$$

Theorem

In $\mathbb{R}^r$, *for* $|A| = n$,

$$\mathcal{C}_{n-r-\delta} \subseteq \mathcal{D}_{n-r-3\delta}$$

That's all!

Thank you!