Bounds on hyperbolicity constant of line graphs.

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VIII Jornadas de Matemática Discreta y Algorítmica.

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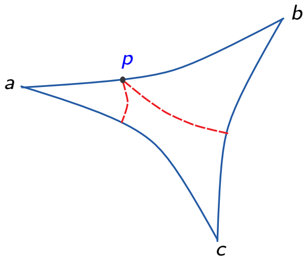
Gromov hyperbolic spaces

Definition

Let X be a geodesic metric space and $x_1, x_2, x_3 \in X$, a geodesic triangle $T = \{x_1, x_2, x_3\}$ is the union of the three geodesics $[x_1x_2]$, $[x_2x_3]$ and $[x_3x_1]$ in X.



T is δ-thin, if any side of T is contained in a δ-neighborhood of the union of the two other sides.

$$d(p, [x_ix_j] \cup [x_jx_k]) \leq \delta \quad \forall p \in [x_ix_k]$$

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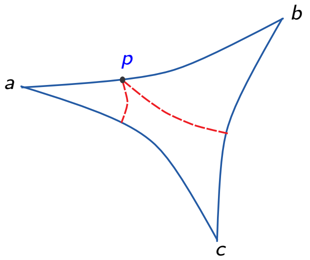

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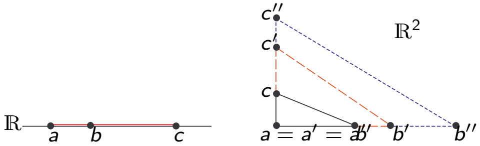
$$d(p, [x_ix_j] \cup [x_jx_k]) \leq \delta \quad \forall p \in [x_ix_k]$$

The space X is δ-hyperbolic (in the Gromov sense), if every geodesic triangle T in X is δ-thin.

Examples of hyperbolic spaces

- $\mathbb{R}^n$ with Euclidean metric is hyperbolic if and only if $n = 1$.



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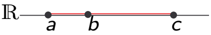
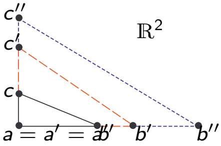
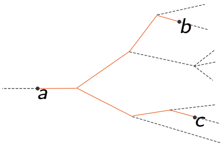




- All $\mathbb{R}$-tree (a tree with edges of arbitrary lengths) is 0-hyperbolic.



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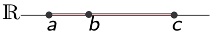

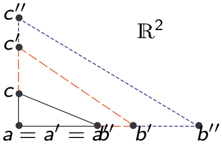

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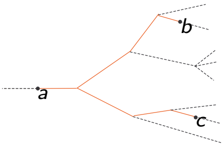

- The open unit disk ($\mathbb{D}$) with the Poincaré metric is $\log\left(1+\sqrt{2}\right)$-hyperbolic.

Open Problem: When do we have a Gromov space?

Let **X** be any geodesic metric space. ¿It is hyperbolic?

Step 1.

We have to consider an arbitrary geodesic triangle **T**

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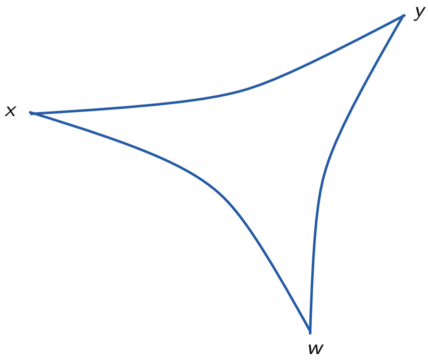

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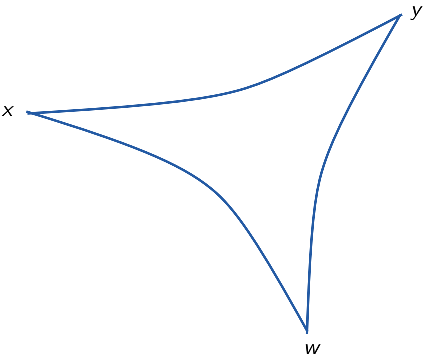
Let $\mathbf{X}$ be any geodesic metric space. ¿It is hyperbolic?

Step 2. *for every* $P \in \mathbf{T}$, *calculate* $\text{dist}(P, \mathbf{A})$ *with* $\mathbf{A}$ *the union of the other two sides of the triangle to which* P *does not belong to.*



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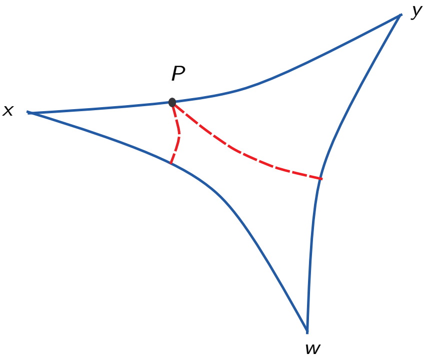

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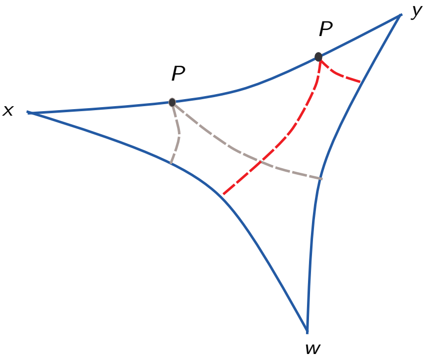

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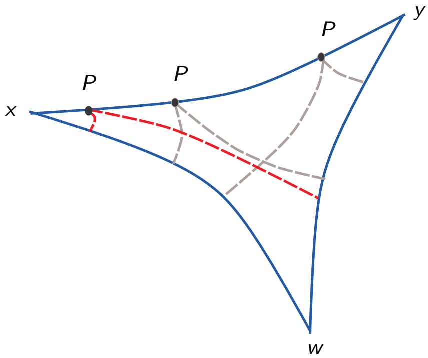

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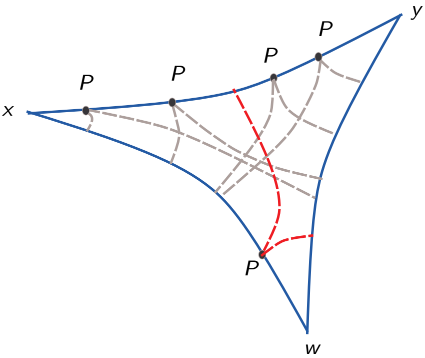

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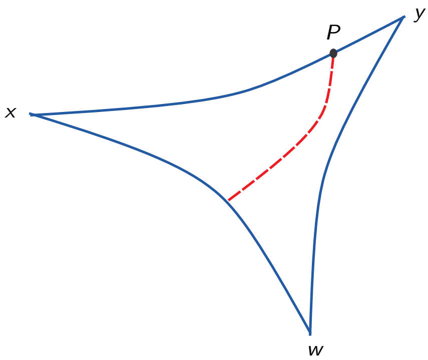
Paso 3. *We have to take* $\delta_{\mathbf{T}} := \max_P (\mathrm{dist}(P, A))$.



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Repeat the steps over all the possible choices for T

$$\delta_{\mathbf{X}} := \sup_{T} \delta_{T}$$

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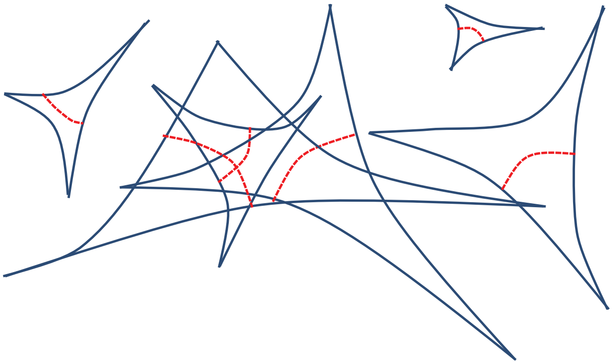

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The hyperbolicity constant of a graph provides a measure of how much a graph resembles a tree.

It is interesting to obtain inequalities relating the hyperbolicity constant and other parameters of graphs. Another natural problem is to study the invariance of the hyperbolicity of graphs under appropriate transformations.

This work.

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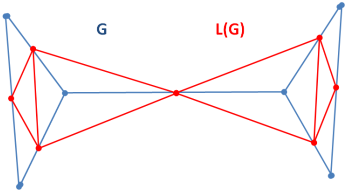
- We obtain some relations between the hyperbolicity constant of the line graph L(G) of G and some natural properties of G such as its girth and its circumference.
- If $\{G_n\}$ is a T-decomposition of G, the line graph $\mathcal{L}(G)$ is hyperbolic if and only if $\sup_n \delta(\mathcal{L}(G_n))$ is finite.
- We characterize the graphs G with $\delta(\mathcal{L}(G)) < k$.

Line Graph

Definition

The line graph $\mathcal{L}(G)$ of a graph G is a graph which has a vertex $V_{e_i} \in V(\mathcal{L}(G))$ for each edge e_i of G, and an edge joining V_{e_i} and V_{e_j} when $e_i \cap e_j \neq \varnothing$.



Some authors define the edges of line graph with length 1 or another fixed constant (k), but we also define the length of the edge $[V_{e_i}, V_{e_j}] \in E(\mathcal{L}(G))$ as $(L(e_i) + L(e_j))/2$.

Results

Theorem

Let G be any graph such that every edge has length k. Then there exists a $(k/2)$-full $(1, k)$-quasi-isometry from G on its line graph $\mathcal{L}(G)$ and, consequently,

$$G \text{ is hyperbolic if and only if } \mathcal{L}(G) \text{ is hyperbolic.}$$

Furthermore, if G (respectively, $\mathcal{L}(G)$) is δ-hyperbolic, then $\mathcal{L}(G)$ (respectively, G) is δ'-hyperbolic, where δ' is a constant which just depends on δ and k.

Results

Theorem

For any graph G which every edge has length k, we have

$$\frac{1}{12}\,\delta(G) - \frac{3k}{4} \leq \delta(\mathcal{L}(G)) \leq 12\,\delta(G) + 18k.$$

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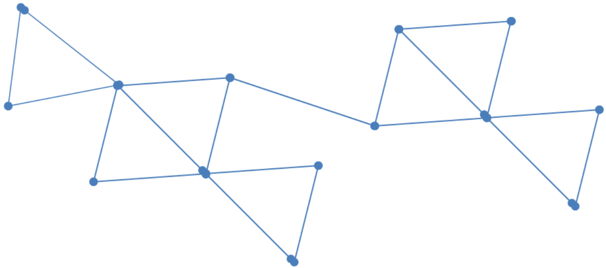
Theorem

If G is a graph with n vertices $v_1, \ldots, v_n$, then

$$\delta(\mathcal{L}(G)) + \delta(G) \leq \frac{k}{8}\sum_{i=1}^{n}(\deg_G(v_i))^2.$$

T-decompositions



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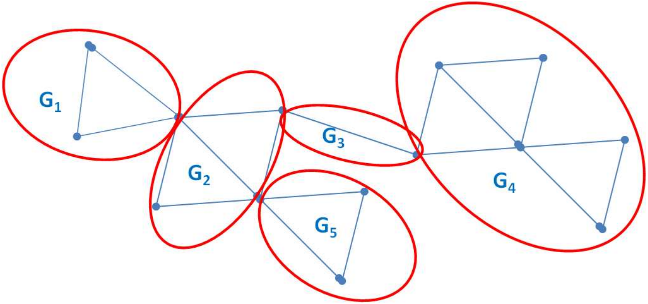

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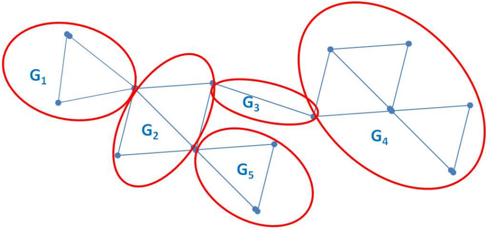

We say that the family of subgraphs $\{G_n\}_n$ of G is a *T-decomposition* of G if the graph R is a tree.

Theorem

If $\{G_n\}_n$ is any T-decomposition of any graph G, then

$$\sup_n \delta(\mathcal{L}(G_n)) \leq \delta(\mathcal{L}(G)) \leq \sup_n \delta(\mathcal{L}(G_n)) + k\,.$$

Results

Theorem

If G is any graph with $\delta(\mathcal{L}(G)) < k$, then there are just two possibilities: $\delta(\mathcal{L}(G)) = 0$ or $\delta(\mathcal{L}(G)) = 3k/4$. Furthermore,

- $\delta(\mathcal{L}(G)) = 0$ *if and only if G is a tree with maximum degree $\Delta \leq 2$,*
- $\delta(\mathcal{L}(G)) = 3k/4$ *if and only if G is either a tree with maximum degree $\Delta = 3$ or isomorphic to C_3.*

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Theorem

If G is any graph with $\delta(G) < k$, then $\delta(\mathcal{L}(G)) \leq 7k/4$.

Graphs with edges of arbitrary lengths

We define a function $h : PM_{\mathcal{L}}V(\mathcal{L}(G)) \longrightarrow PMV(G)$

Lemma

For every $x, y \in h(\mathcal{L}(G))$, we have

$$d_G(x, y) = d_{\mathcal{L}(G)}(h^{-1}(x), h^{-1}(y)).$$

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Proposition

For every $x, y \in \mathcal{L}(G)$ we have

$$d_{\mathcal{L}(G)}(x,y) - 2l_{max} \leq d_G(h(x), h(y)) \leq d_{\mathcal{L}(G)}(x,y),$$

with $l_{max} = \sup_{e \in E(G)} L(e)$.

Graphs with edges of arbitrary lengths

Theorem

Let G be a graph and consider $\mathcal{L}(G)$ the line graph of G. Then

$$\delta(G) \leq \delta(\mathcal{L}(G)) \leq 5\delta(G) + 3l_{max},$$

with $l_{max} = \sup_{e \in E(G)} L(e)$.

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with $l_{max} = \sup_{e \in E(G)} L(e)$.

Corollary

Let G be any graph such that every edge has length k and consider $\mathcal{L}(G)$ the line graph of G. Then

$$\delta(G) \leq \delta(\mathcal{L}(G)) \leq 5\delta(G) + \frac{5k}{2}.$$

Thanks for your attention.

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