Green function on product networks

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Motivation
Motivation

\[
\begin{cases}
\mathcal{L}_q(u) = f & \text{on } F \\
u = g & \text{on } \delta(F)
\end{cases}
\]
Our objective
Our objective

Determine the Green function of a *product network* in terms of
Our objective

\[ \text{\LARGE →} \quad \text{Determine the Green function of a product network in terms of } \Gamma_1 \text{ Green functions} \]
Determine the Green function of a *product network* in terms of

$\Gamma_1$ Green functions

$\Gamma_2$ Schrödinger operator’s eigenvalues and eigenfunctions
Our objective

\[ \text{Determine the Green function of a } \textit{product network} \text{ in terms of } \Gamma_1 \text{ Green functions} \]

\[ \Gamma_2 \text{ Schrödinger operator’s eigenvalues and eigenfunctions} \]

\[ \text{Determine the Green function of a } \textit{spider network} \text{ in terms of} \]
Our objective

\[\rightarrow\] Determine the Green function of a *product network* in terms of

\[\Gamma_1\] Green functions

\[\Gamma_2\] Schrödinger operator’s eigenvalues and eigenfunctions

\[\rightarrow\] Determine the Green function of a *spider network* in terms of

\[\text{cycles}\] Green functions
Our objective

~~~ Determine the Green function of a product network in terms of

\[ \Gamma_1 \]
Green functions

\[ \Gamma_2 \]
Schrödinger operator’s eigenvalues and eigenfunctions

~~~ Determine the Green function of a spider network in terms of

cycles \quad Green functions

paths \quad Schrödinger operator’s eigenvalues and eigenfunctions
Notations and basic results
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\[ \Gamma = (V, c) \] network
Notations and basic results

\[ \Gamma = (V, c) \quad \text{network} \]

\[ F \text{ proper subset of } V, \quad \delta(F) \text{ boundary of } F \]
Notations and basic results

\[ \Gamma = (V, c) \quad \text{network} \]

\[ F \text{ proper subset of } V, \quad \delta(F) \text{ boundary of } F \]

\[ \mathcal{L} : \mathcal{C}(\overline{F}) \longrightarrow \mathcal{C}(\overline{F}) \quad \text{Laplacian of } \Gamma \]
Notations and basic results

$\Gamma = (V, c)$ network

$F$ proper subset of $V$, $\delta(F)$ boundary of $F$

$L : \mathcal{C}(\overline{F}) \longrightarrow \mathcal{C}(\overline{F})$ Laplacian of $\Gamma$ $u \in \mathcal{C}(\overline{F})$

$x \in F \quad \mapsto \quad L(u)(x) = \sum_{y \in \overline{F}} c(x, y)(u(x) - u(y))$

$x \in \delta(F) \quad \mapsto \quad L(u)(x) = \sum_{y \in \overline{F}} c(x, y)(u(x) - u(y)) = \partial u / \partial n$
Notations and basic results

\( \Gamma = (V, c) \) \quad \text{network}

\( F \) proper subset of \( V \), \( \delta(F) \) boundary of \( F \)

\( \mathcal{L} : \mathcal{C}(\overline{F}) \rightarrow \mathcal{C}(\overline{F}) \) \quad \text{Laplacian of} \ \Gamma \quad u \in \mathcal{C}(\overline{F})

\[ x \in F \quad \implies \quad \mathcal{L}(u)(x) = \sum_{y \in \overline{F}} c(x, y)(u(x) - u(y)) \]

\[ x \in \delta(F) \quad \implies \quad \mathcal{L}(u)(x) = \sum_{y \in F} c(x, y)(u(x) - u(y)) \]
Notations and basic results

\( \Gamma = (V, c) \) network

\( F \) proper subset of \( V \), \( \delta(F) \) boundary of \( F \)

\( \mathcal{L} : \mathcal{C}(\overline{F}) \rightarrow \mathcal{C}(\overline{F}) \) Laplacian of \( \Gamma \) \( u \in \mathcal{C}(\overline{F}) \)

\[ \begin{align*}
x \in F & \quad \implies \quad \mathcal{L}(u)(x) = \sum_{y \in \overline{F}} c(x, y)(u(x) - u(y)) \\
x \in \delta(F) & \quad \implies \quad \mathcal{L}(u)(x) = \sum_{y \in F} c(x, y)(u(x) - u(y)) = \frac{\partial u}{\partial n_F}(x)
\end{align*} \]

normal derivative
Notations and basic results

\[ \mathcal{L}_q(u) = \mathcal{L}(u) + qu \quad \text{Schrödinger operator of } \Gamma \]
Notations and basic results

\[ \mathcal{L}_q(u) = \mathcal{L}(u) + qu \quad \text{Schrödinger operator of } \Gamma \]

\[ \omega \in \mathcal{C}(\overline{F}) \quad \text{weight on } \overline{F} \quad \iff \quad \sum_{x \in \overline{F}} \omega^2(x) = 1 \]
Notations and basic results

\[ \mathcal{L}_q(u) = \mathcal{L}(u) + qu \quad \text{Schrödinger operator of } \Gamma \]

\[ \omega \in \mathcal{C}(F) \quad \text{weight on } F \quad \Leftrightarrow \quad \sum_{x \in F} \omega^2(x) = 1 \]

\[ q_\omega = -\omega^{-1} \mathcal{L}(\omega) \quad \text{Potential given by } \omega \]
Notations and basic results

\[ \mathcal{L}_q(u) = \mathcal{L}(u) + qu \quad \text{Schrödinger operator of } \Gamma \]

\[ \omega \in \mathcal{C}(\overline{F}) \quad \text{weight on } \overline{F} \quad \Leftrightarrow \quad \sum_{x \in \overline{F}} \omega^2(x) = 1 \]

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Lemma (Bendito, Carmona, Encinas 2005)

\[ \mathcal{L}_q \text{ positive definite on } \mathcal{C}(F) \quad \Leftrightarrow \quad \text{there exists a weight } \omega \text{ such that } q \geq q_\omega \]
Notations and basic results

\[ \mathcal{L}_q(u) = \mathcal{L}(u) + qu \quad \text{Schrödinger operator of } \Gamma \]

\[ \omega \in \mathcal{C}(\overline{F}) \quad \text{weight on } \overline{F} \quad \iff \sum_{x \in \overline{F}} \omega^2(x) = 1 \]

\[ q_\omega = -\omega^{-1} \mathcal{L}(\omega) \quad \text{Potential given by } \omega \]

Lemma (Bendito, Carmona, Encinas 2005)

\[ \mathcal{L}_q \text{ positive definite on } \mathcal{C}(F) \iff \text{there exists a weight } \omega \text{ such that } q \geq q_\omega \]

\[ \Downarrow \quad \text{We work with potentials of the form } q = q_\omega + \lambda, \text{ where } \lambda > 0 \]
Product networks
Product networks

\[ \Gamma_1 \]

\[ \Gamma_2 \]
Product networks

\[ \Gamma_1 \times \Gamma_2 \]
Product networks - vertices

\[ V(\Gamma_1) = \{x_1^1, \ldots, x_n^1\} \]

\[ V(\Gamma_2) = \{x_1^2, \ldots, x_m^2\} \]
Product networks - vertices

\[ V(\Gamma_1) = \{x_1^1, \ldots, x_n^1\} \]

\[ V(\Gamma_2) = \{x_1^2, \ldots, x_m^2\} \]

\[ V(\Gamma_1 \times \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2) \]

\[ x_{ji} = (x_j^1, x_i^2) \]
Product networks - conductances

\[ \Gamma_1 \times \Gamma_2 \]

\( c_1 \)'s

\( c_2 \)'s
Product networks - conductances

\[ \Gamma_1 \times \Gamma_2 \]

\[ c(x_{ji}, x_{lk}) = \begin{cases} 
  c_1(x^1_{ji}, x^1_{lj}) & \text{if } i = k \\
  c_2(x^2_{xi}, x^2_{zk}) & \text{if } l = j \\
  0 & \text{otherwise}
\end{cases} \]
Product networks - interior and boundary sets

\[
\Gamma_1 \times \Gamma_2
\]

\[
F_1 \times F_2
\]
Product networks - interior and boundary sets

\[ \Gamma_1 \times \Gamma_2 \]

\[ \begin{array}{c}
\Gamma_1 \\
F_1 \\
\end{array} \]

\[ \begin{array}{c}
\Gamma_2 \\
F_2 \\
\end{array} \]

\[ \begin{aligned}
&x_{1m} \\
&x_{2m} \\
&x_{3m} \\
&x_{4m} \\
\end{aligned} \]

\[ \begin{aligned}
&x_{11} \\
&x_{21} \\
&x_{31} \\
&x_{41} \\
\end{aligned} \]
Product networks - interior and boundary sets

\[ \Gamma_1 \times \Gamma_2 \]

**Gammas**
- \( \Gamma_1 \)
- \( \Gamma_2 \)

**F's**
- \( F_1 \)
- \( F_2 \)

**H's**
- \( H_1 \)
- \( H_m \)

**Points**
- \( x_{11} \)
- \( x_{21} \)
- \( x_{31} \)
- \( x_{41} \)
- \( x_{1m} \)
- \( x_{2m} \)
- \( x_{3m} \)
- \( x_{4m} \)

**Subscripts**
- \( x_{11} \)
- \( x_{21} \)
- \( x_{31} \)
- \( x_{41} \)
- \( x_{1m} \)
- \( x_{2m} \)
- \( x_{3m} \)
- \( x_{4m} \)
Product networks - functions

\[ u_1 \in \mathcal{C}(V(\Gamma_1)) \quad \text{and} \quad u_2 \in \mathcal{C}(V(\Gamma_2)) \]

\[ u_1(x^1_j) = a \quad \text{and} \quad u_2(x^2_i) = b \]

\[ \Gamma_1 \times \Gamma_2 \]
Product networks - functions

\[ u_1 \in \mathcal{C}(V(\Gamma_1)) \]

\[ u_2 \in \mathcal{C}(V(\Gamma_2)) \]

\[ u_1(x_j^1) = a \]

\[ u_2(x_i^2) = b \]

\[ (u_1 \otimes u_2)(x_{ji}) = u_1(x_j^1) \ u_2(x_i^2) = ab \]

\[ u_1 \otimes u_2 \in \mathcal{C}(V(\Gamma_1 \times \Gamma_2)) \]
**Remark**

*Given two weights* $\omega_1 \in C(F_1)$ *and* $\omega_2 \in C(\overline{F}_2)$ *then*

\[ \omega_1 \otimes \omega_2 \]

*is a weight on* $\overline{F}$. 
Remark

Given two weights \( \omega_1 \in C(F_1) \) and \( \omega_2 \in C(F_2) \) then

\[
\omega_1 \otimes \omega_2
\]

is a weight on \( F \).

\[
\langle \cdot, \cdot \rangle_{\omega_2} : C(F_2) \times C(F_2) \longrightarrow \mathbb{R} \quad \text{inner product w. r. to } \omega_2 \in C(F_2)
\]

\[
u_2, v_2 \in C(F_2) \implies \langle u_2, v_2 \rangle_{\omega_2} = \sum_{r=1}^{m} u_2(x_r^2)v_2(x_r^2)\omega_2(x_r^2)
\]
Lemma

Given an orthonormal basis \( \{ \phi_r \}_{r=2}^{m-1} \) on \( C(F_2) \) w.r. to \( \omega_2 \) and \( f \in (F) \) then

\[
f = \sum_{r=2}^{m-1} f_r \otimes \phi_r \quad \left( f_r(x_j^1) = \sum_{s=1}^{m} f(x_{js}) \phi_r(x_s^2) \omega_2(x_s^2) \quad \forall x_j^1 \in F_1 \right)
\]
Lemma

Given an orthonormal basis \( \{ \phi_r \}_{r=2}^{m-1} \) on \( C(F_2) \) w.r. to \( \omega_2 \) and \( f \in (F) \) then

\[
f = \sum_{r=2}^{m-1} f_r \otimes \phi_r \quad \left( f_r(x^1_j) = \sum_{s=1}^{m} f(x_{js}) \phi_r(x^2_s) \omega_2(x^2_s) \ \forall x^1_j \in F_1 \right)
\]

In addition, \( f = g \iff f_r = g_r \) for all \( r = 2, \ldots, m - 1 \)
Lemma (Bendito, Carmona, Encinas 2005)

There exists a set of real values $\lambda_2 \leq \ldots \leq \lambda_{m-1}$ and an orthonormal basis $\{\phi_r\}_{r=2}^{m-1} \subset C(F_2)$ w.r. to $\omega_2$ s.t.

- $\mathcal{L}_{q\omega_2}^2(\phi_r) = \lambda_r \phi_r$ on $F_2$ and $\phi_r(x_1^2) = \phi_r(x_m^2) = 0$
Lemma (Bendito, Carmona, Encinas 2005)

There exists a set of real values $\lambda_2 \leq \ldots \leq \lambda_{m-1}$ and an orthonormal basis $\{\phi_r\}_{r=2}^{m-1} \subset \mathcal{C}(F_2)$ w.r. to $\omega_2$ s.t.

- $L_{q\omega_2}^2(\phi_r) = \lambda_r \phi_r$ on $F_2$ and $\phi_r(x_1^2) = \phi_r(x_m^2) = 0$

- $\mu \in \mathbb{R}$ eigenvalue of $L_{q\omega_2}^2$ $\Rightarrow$ $\mu = \lambda_r$ for some $r \in \{2, \ldots, m-1\}$
Lemma (Bendito, Carmona, Encinas 2005)

There exists a set of real values $\lambda_2 \leq \ldots \leq \lambda_{m-1}$ and an orthonormal basis $\{\phi_r\}_{r=2}^{m-1} \subset \mathcal{C}(F_2)$ w.r. to $\omega_2$ s.t.

- $\mathcal{L}_{q\omega_2}^2(\phi_r) = \lambda_r \phi_r$ on $F_2$ and $\phi_r(x_1^2) = \phi_r(x_m^2) = 0$
- $\mu \in \mathbb{R}$ eigenvalue of $\mathcal{L}_{q\omega_2}^2 \Rightarrow \mu = \lambda_r$ for some $r \in \{2, \ldots, m - 1\}$
- $\mathcal{L}_{q\omega_2}^2(v_2) = \sum_{r=2}^{m-1} \lambda_r \langle v_2, \phi_r \rangle_{\omega_2} \phi_r$
Green operator on product networks
Green operator on product networks

\[ \omega_1 \text{ weight on } F_1, \quad \omega_2 \text{ weight on } \overline{F}_2 \]
Green operator on product networks

\( \omega_1 \) weight on \( F_1 \), \( \omega_2 \) weight on \( \overline{F_2} \)

we assume to know

\[ \Gamma_1 \]

the Green functions on \( \Gamma_1 \)

with respect to \( \omega_1 \)

and a real positive value
Green operator on product networks

\[ \omega_1 \text{ weight on } F_1, \quad \omega_2 \text{ weight on } \overline{F}_2 \]

we assume to know

\[ \lambda_2 \leq \ldots \leq \lambda_{m-1} \] eigenvalues and

\[ \{\phi_r\}_{r=2}^{m-1} \] eigenfunctions of \( L^2_{q\omega_2} \) with

\[ \phi_r(x_1^2) = \phi_r(x_m^2) = 0 \] forming an orthonormal basis on \( \mathcal{C}(F_2) \)

the Green functions on \( \Gamma_1 \)
with respect to \( \omega_1 \)
and a real positive value
Green operator on product networks

$x_{lk} \in F$ interior vertex of the product network
Green operator on product networks

\( x_{lk} \in F \) \hspace{1cm} interior vertex of the product network

\( \lambda > 0 \) real value
Green operator on product networks

\[ x_{lk} \in F \quad \text{interior vertex of the product network} \]

\[ \lambda > 0 \quad \text{real value} \]

**Proposition (Bendito, Carmona, Encinas 2005)**

The boundary-value problem on the product network

\[
\begin{cases}
\mathcal{L}_q(u) = \varepsilon_{x_{lk}} & \text{on } F \\
\quad u = 0 & \text{on } \delta(F)
\end{cases}
\]

has a unique solution, known as the Green operator with pole on \( x_{lk} \)

\[ u = \mathcal{G}_q(\varepsilon_{x_{lk}}) \]
Green operator on product networks

\[ G_q(x, x_{lk}) = G_q(\varepsilon x_{lk})(x) \text{ for all } x \in \overline{F} \]  
Green kernel
Green operator on product networks

\[ G_q(x, x_{lk}) = G_q(\varepsilon_{x_{lk}})(x) \text{ for all } x \in \overline{F} \]

Green kernel

**Proposition**

The Schrödinger operator of the product network can be written in terms of its factors as

\[ \mathcal{L}_q(u_1 \otimes u_2)(x_{ji}) = \left( \mathcal{L}^1_{q_1}(u_1) \otimes u_2 \right)(x_{ji}) + \left( u_1 \otimes \mathcal{L}^2_{q_{\omega_2}}(u_2) \right)(x_{ji}) \]

where \( q = q_{\omega_1} \otimes \omega_2 + \lambda \) and \( q_1 = q_{\omega_1} + \lambda \)
Theorem

The solution of problem

\[
\begin{cases}
\mathcal{L}_q(u) = \varepsilon x_{lk} & \text{on } F \\
u = 0 & \text{on } \delta(F)
\end{cases}
\]

is given by

\[
u = \sum_{r=2}^{m-1} u_r \otimes \phi_r & \text{on } \mathring{F}
\]

where \( u_r(x^{\frac{1}{j}}) = \phi_r(x^{\frac{2}{k}})\omega_2(x^{\frac{2}{k}})G^{1}_{qr}(x^{\frac{1}{j}}, x^{\frac{1}{l}}) \) and \( q_r = q\omega_1 + \lambda + \lambda_r \)
Green operator on product networks

Theorem

The solution of problem

\[
\begin{cases}
\mathcal{L}_q(u) = \varepsilon x_{lk} & \text{on } F \\
u = 0 & \text{on } \delta(F)
\end{cases}
\]

is given by

\[
u = \sum_{r=2}^{m-1} u_r \otimes \phi_r & \text{on } F
\]

where \( u_r(x^1_j) = \phi_r(x^2_k)\omega_2(x^2_k)G^1_{qr}(x^1_j, x^1_l) \) and \( q_r = q\omega_1 + \lambda + \lambda_r \)

Therefore,

\[
G_q(x_{ji}, x_{lk}) = \omega_2(x^2_k) \sum_{r=2}^{m-1} \phi_r(x^2_k)\phi_r(x^2_l)G^1_{qr}(x^1_j, x^1_l)
\]
Spider networks
Spider networks

\[ v^S_j = x^S_{j \tilde{m} + 1} \]

\[ v^S_i \]

\[ v^S_n \]

\[ v^S_2 \]

\[ v^S_3 \]

\[ v^S_1 \]

radius 1

radius 2

radius 3

radius \( n \)

radius \( j \)

circle \( \tilde{m} \)

circle \( i \)

circle 1

Araúz, Carmona, Encinas (UPC)

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Green function on product networks

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Spider networks

product network

spider network

removal of all the boundary edges,
identification of all the vertices of $H_1$

radius j

circle $i$

radius $m$

radius $n$

circle $m$

radius $j$

$v_j^S = x_j^S \tilde{m} + 1$
Spider networks - interior and boundary

\[ \delta(F') \]

\[ F \]

\[ \{v_0, v_1, v_2, v_3, \ldots, v_n\} \]

\[ x_{00} \]

\[ \{v_1, v_2, v_3, \ldots, v_n\} \]
Spider networks - conductances and functions

conductances remain the same as in the product

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Green function on product networks
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a function \( u_S \in \mathcal{C}(\overline{\mathcal{F}_S}) \) on the spider can be adapted as a function \( u \in \mathcal{C}(\overline{\mathcal{F}}) \) on the product by taking the same values on \( \overline{\mathcal{F}} \setminus \{x_{00}^S\} \) and \( u(x_{j1}) = u_S(x_{00}^S) \) on \( H_1 \).
a function $u_S \in \mathcal{C}(\overline{F}_S)$ on the spider can be adapted as a function $u \in \mathcal{C}(\overline{F})$ on the product by taking the same values on $\overline{F} \setminus \{x_{00}^S\}$ and $u(x_{j1}) = u_S(x_{00}^S)$ on $H_1$.

A weight $\omega_S$ in $\overline{F}_S$ does not adapt as a weight on the product!

We take $\sigma$ s. t. $\overline{\omega} = \sigma^{-1}\omega$ on $\overline{F}$ is a weight on the product.
Green operator on spider networks
Green operator on spider networks

\[ \omega_S \text{ weight on } \overline{F}_S, \quad \lambda > 0 \text{ real value} \]
Green operator on spider networks

\( \omega_S \) weight on \( \overline{F}_S \), \( \lambda > 0 \) real value

**Proposition**

**Schrödinger operator in terms of a product network**

\[
\mathcal{L}^S_{q_S}(u_S) = \mathcal{L}_q(u)
\]

\[
\mathcal{L}^S_{q_S}(u_S)(x_{00}^S) = \left( \lambda + \frac{1}{\omega(x_{11})} \sum_{t=1}^{n} c(x_{t1}, x_{t2}) \omega(x_{t2}) \right) u(x_{11}) \\
- \sum_{t=1}^{n} c(x_{t1}, x_{t2}) u(x_{t2})
\]

where \( q_S = q_{\omega_S} + \lambda \) and \( q = q_{\omega} + \lambda \)
Green operator on spider networks

\( x^S_{lk} \in F_S \) interior vertex of the spider network
Green operator on spider networks

\[ x_{lk}^{S} \in F_{S} \quad \text{interior vertex of the spider network} \]

**Proposition**

The boundary-value problem on the spider network

\[
\begin{cases}
\mathcal{L}^{S}_{qS}(u_{S}) = \varepsilon_{x_{lk}^{S}} & \text{on } F_{S} \\
 u_{S} = 0 & \text{on } \delta(F_{S})
\end{cases}
\]

has a unique solution, known as the **Green operator of the spider network with pole on** \( x_{lk} \)

\[ u_{S} = G^{S}_{qS}(\varepsilon_{x_{lk}^{S}}) \]
Green operator on spider networks

\[ x^S_{lk} \in F_S \text{ interior vertex of the spider network} \]

**Proposition**

The boundary-value problem on the spider network

\[
\begin{align*}
\mathcal{L}^S_{qs}(u^S) = \varepsilon^S_{x_{lk}} & \quad \text{on } F_S \\
u^S = 0 & \quad \text{on } \delta(F_S)
\end{align*}
\]

It is given by

\[
u^S = \mathcal{G}_q(v\chi_F) + \alpha \sum_{t=1}^{n} c(x_{t1}, x_{t2})\mathcal{G}_q(\varepsilon_{xt2}) + \alpha \chi_{H1}
\]

on \( F_S \), where \( \alpha \) is a constant.
Gracias!