Green function on product networks

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Motivation

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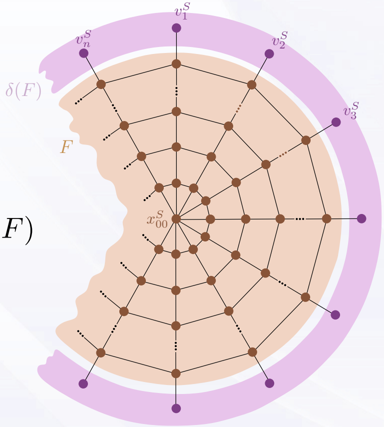
$$\begin{cases} \mathcal{L}_q(u) = f & \text{on } F \\ u = g & \text{on } \delta(F) \end{cases}$$



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Notations and basic results

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$$x \in \delta(F) \quad \rightsquigarrow \quad \mathcal{L}(u)(x) = \sum_{y \in F} c(x,y)\big(u(x) - u(y)\big) = \frac{\partial u}{\partial \mathsf{n}_F}(x)$$

normal derivative

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$\mathcal{L}_q$ positive definite on $\mathcal{C}(F)$ $\Leftrightarrow$ there exists a weight ω such that $q \geq q_\omega$

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$\mathcal{L}_q$ positive definite on $\mathcal{C}(F)$ $\Leftrightarrow$ there exists a weight ω such that $q \geq q_\omega$

⇝ We work with potentials of the form $q = q_\omega + \lambda$, where $\lambda > 0$

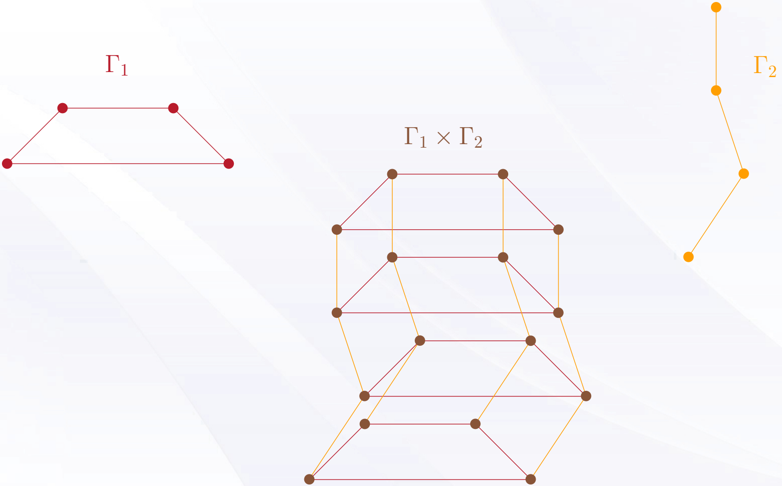
Product networks

Product networks

Γ_1

Γ_2

Product networks



Product networks - vertices

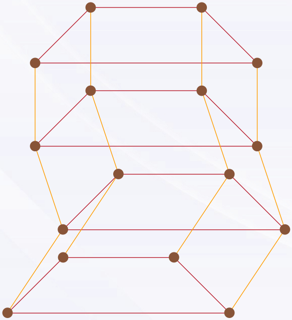
$V(\Gamma_1) = \{x^1_1, \dots, x^1_n\}$

Γ_1

$V(\Gamma_2) = \{x^2_1, \dots, x^2_m\}$

Γ_2

$\Gamma_1 \times \Gamma_2$



Product networks - vertices

$V(\Gamma_1) = \{x_1^1, \dots, x_n^1\}$

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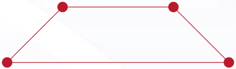
Γ_1

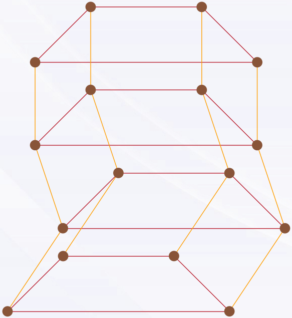


x_j^1

Γ_2

x_i^2

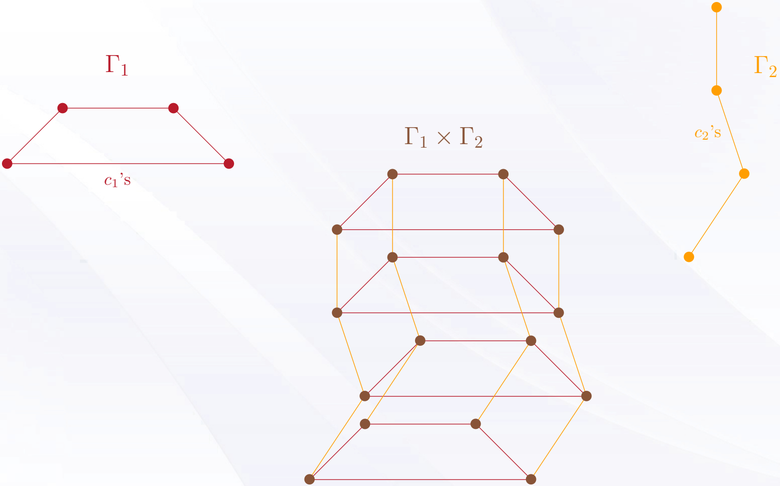
$\Gamma_1 \times \Gamma_2$



$x_{ji} = (x_j^1, x_i^2)$

$V(\Gamma_1 \times \Gamma_2) = V(\Gamma_1) \times V(\Gamma_2)$

Product networks - conductances



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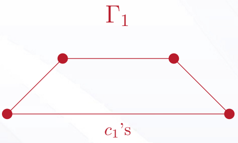
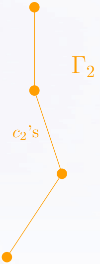
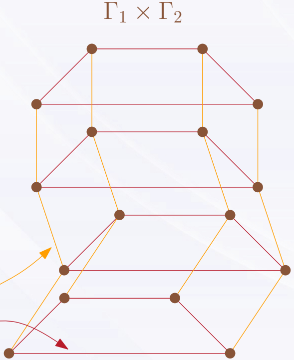
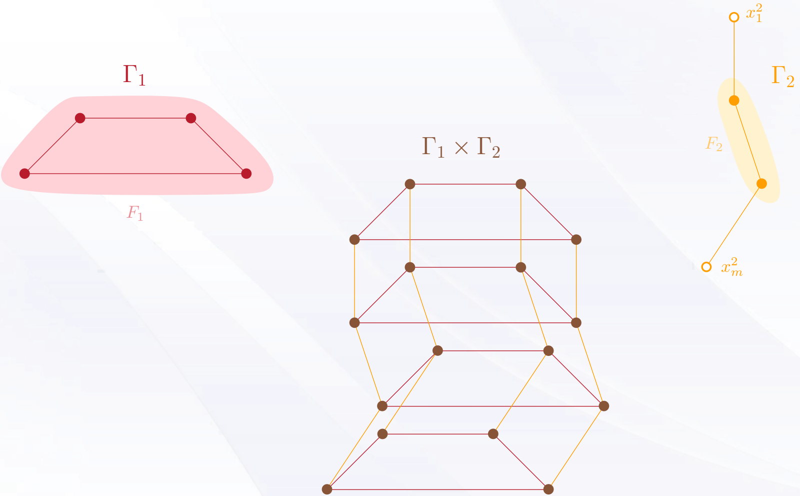






$$c(x_{ji}, x_{lk}) = \begin{cases} c_1(x_j^1, x_l^1) & \text{if } i = k \\ c_2(x_i^2, x_k^2) & \text{if } l = j \\ 0 & \text{otherwise} \end{cases}$$

Product networks - interior and boundary sets



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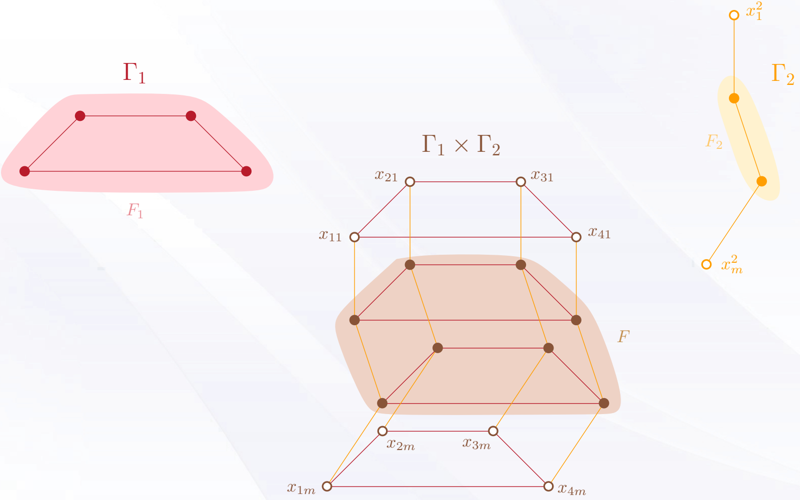

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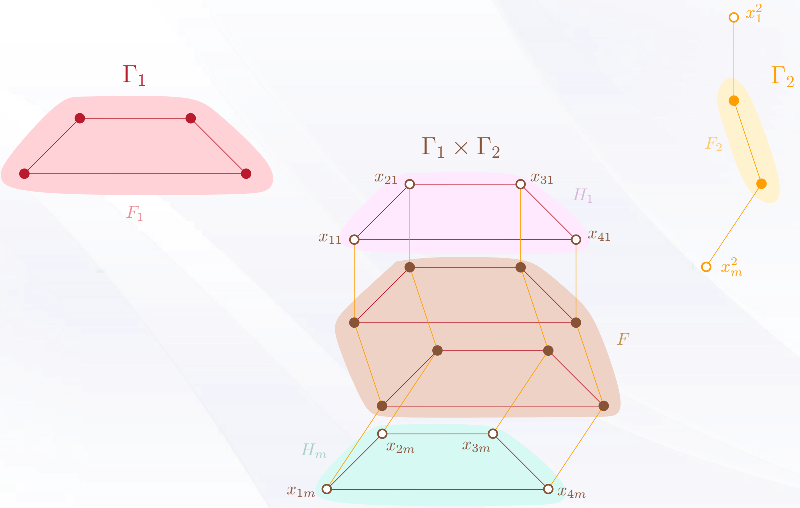
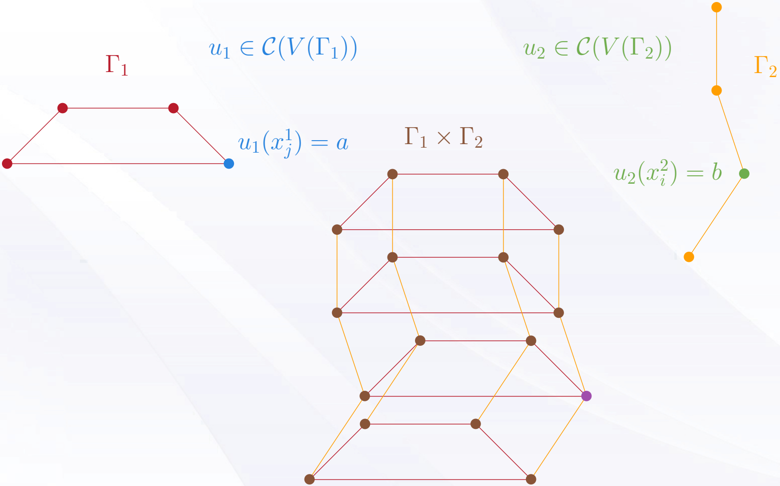


Product networks - functions



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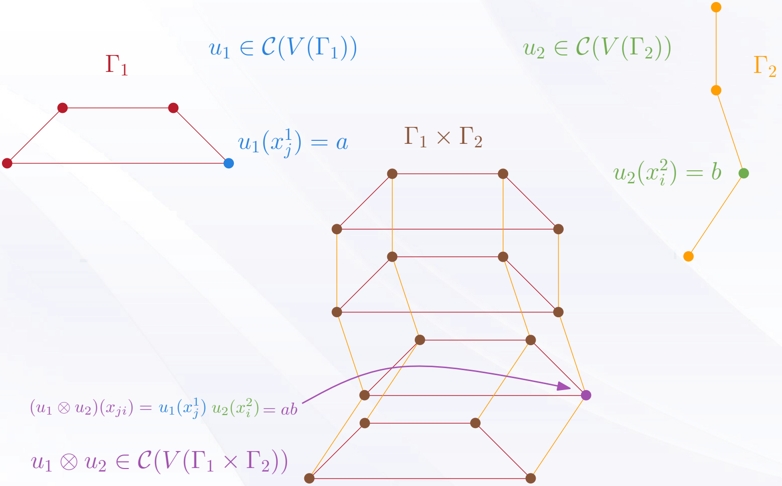

Product networks - functions

Remark

Given two weights $\omega_1 \in \mathcal{C}(F_1)$ *and* $\omega_2 \in \mathcal{C}(\overline{F}_2)$ *then*

$$\omega_1 \otimes \omega_2$$

is a weight on $\overline{F}$.

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$\langle \cdot, \cdot \rangle_{\omega_2} : \mathcal{C}(\overline{F}_2) \times \mathcal{C}(\overline{F}_2) \longrightarrow \mathbb{R}$ inner product w. r. to $\omega_2 \in \mathcal{C}(\overline{F}_2)$

$$u_2, v_2 \in \mathcal{C}(\overline{F}_2) \quad \rightsquigarrow \quad \langle u_2, v_2 \rangle_{\omega_2} = \sum_{r=1}^{m} u_2(x_r^2) v_2(x_r^2) \omega_2(x_r^2)$$

Product networks - eigenvalues on Γ_2

Lemma

Given an orthonormal basis $\{\phi_r\}_{r=2}^{m-1}$ *on* $\mathcal{C}(F_2)$ *w.r. to* ω_2 *and* $f \in (F)$ *then*

$$f = \sum_{r=2}^{m-1} f_r \otimes \phi_r \quad \left(f_r(x_j^1) = \sum_{s=1}^{m} f(x_{js})\phi_r(x_s^2)\omega_2(x_s^2) \;\; \forall x_j^1 \in F_1 \right)$$

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In addition, $f = g \Leftrightarrow f_r = g_r$ *for all* $r = 2, \ldots, m-1$

Product networks - eigenvalues on Γ_2

Lemma (Bendito, Carmona, Encinas 2005)

There exists a set of real values $\lambda_2 \leq \ldots \leq \lambda_{m-1}$ *and an orthonormal basis* $\{\phi_r\}_{r=2}^{m-1} \subset \mathcal{C}(F_2)$ *w.r. to* ω_2 *s.t.*

- $\mathcal{L}^2_{q_{\omega_2}}(\phi_r) = \lambda_r \phi_r$ *on* F_2 *and* $\phi_r(x^2_1) = \phi_r(x^2_m) = 0$

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- $\mu \in \mathbb{R}$ *eigenvalue of* $\mathcal{L}^2_{q_{\omega_2}} \Rightarrow \mu = \lambda_r$ *for some* $r \in \{2, \ldots, m-1\}$

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- $\mu \in \mathbb{R}$ *eigenvalue of* $\mathcal{L}^2_{q_{\omega_2}} \Rightarrow \mu = \lambda_r$ *for some* $r \in \{2, \ldots, m-1\}$
- $\mathcal{L}^2_{q_{\omega_2}}(v_2) = \displaystyle\sum_{r=2}^{m-1} \lambda_r \langle v_2, \phi_r \rangle_{\omega_2} \phi_r$

Green operator on product networks

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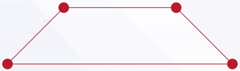
ω_1 weight on F_1, $\quad \omega_2$ weight on $\overline{F}_2$

Green operator on product networks

ω_1 weight on F_1, ω_2 weight on $\overline{F}_2$

we assume to know

Γ_1



the Green functions on Γ_1

with respect to ω_1

and a real positive value

Green operator on product networks

ω_1 weight on F_1, $\quad \omega_2$ weight on $\overline{F}_2$

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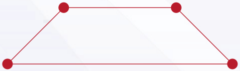
Γ_1



the Green functions on Γ_1

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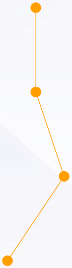
$\lambda_2 \leq \ldots \leq \lambda_{m-1}$ eigenvalues and

$\{\phi_r\}_{r=2}^{m-1}$ eigenfunctions of $\mathcal{L}^2_{q_{\omega_2}}$ with

$\phi_r(x_1^2) = \phi_r(x_m^2) = 0$ forming an

orthonormal basis on $\mathcal{C}(F_2)$



Γ_2

Green operator on product networks

$x_{lk} \in F$ interior vertex of the product network

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$\lambda > 0$ real value

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Proposition (Bendito, Carmona, Encinas 2005)

The boundary-value problem on the product network

$$\begin{cases} \mathcal{L}_q(u) = \varepsilon_{x_{lk}} & \text{on } F \\ u = 0 & \text{on } \delta(F) \end{cases}$$

has a unique solution, known as the Green operator with pole on x_{lk}

$$u = \mathcal{G}_q(\varepsilon_{x_{lk}})$$

Green operator on product networks

$\mathcal{G}_q(x, x_{lk}) = \mathcal{G}_q(\varepsilon_{x_{lk}})(x)$ for all $x \in \overline{F}$ Green kernel

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Proposition

The Schrödinger operator of the product network can be written in terms of its factors as

$$\mathcal{L}_q(u_1 \otimes u_2)(x_{ji}) = \Big(\mathcal{L}^1_{q_1}(u_1) \otimes u_2\Big)(x_{ji}) + \Big(u_1 \otimes \mathcal{L}^2_{q_{\omega_2}}(u_2)\Big)(x_{ji})$$

where $q = q_{\omega_1 \otimes \omega_2} + \lambda$ *and* $q_1 = q_{\omega_1} + \lambda$

Green operator on product networks

Theorem

The solution of problem

$$\begin{cases} \mathcal{L}_q(u) = \varepsilon_{x_{lk}} & \text{on } F \\ u = 0 & \text{on } \delta(F) \end{cases}$$

is given by

$$u = \sum_{r=2}^{m-1} u_r \otimes \phi_r \quad \text{on } \overline{F}$$

where $u_r(x_j^1) = \phi_r(x_k^2)\omega_2(x_k^2)G^1_{q_r}(x_j^1, x_l^1)$ *and* $q_r = q_{\omega_1} + \lambda + \lambda_r$

Green operator on product networks

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Therefore,

$$G_q(x_{ji}, x_{lk}) = \omega_2(x_k^2) \sum_{r=2}^{m-1} \phi_r(x_k^2)\phi_r(x_i^2)G^1_{q_r}(x_j^1, x_l^1)$$

Spider networks

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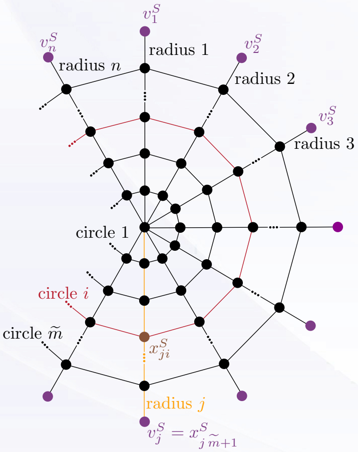
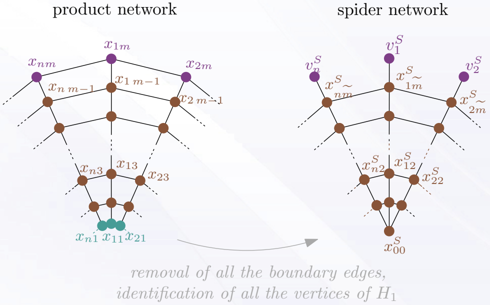
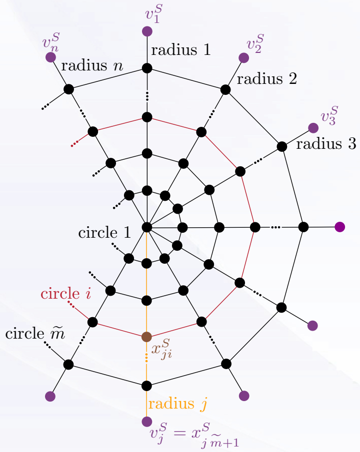
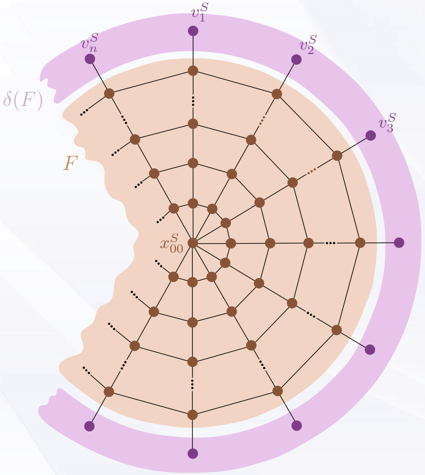


Spider networks



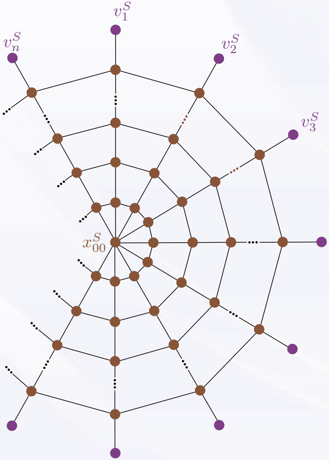


Spider networks - interior and boundary

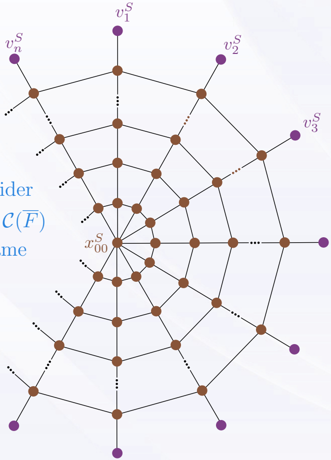


Spider networks - conductances and functions

conductances remain
the same
as in the product



Spider networks - conductances and functions



a function $u_S \in \mathcal{C}(\overline{F}_S)$ on the spider can be adapted as a function $u \in \mathcal{C}(\overline{F})$ on the product by taking the same values on $\overline{F} \setminus \{x_{00}^S\}$ and $u(x_{j1}) = u_S(x_{00}^S)$ on H_1

Spider networks - conductances and functions

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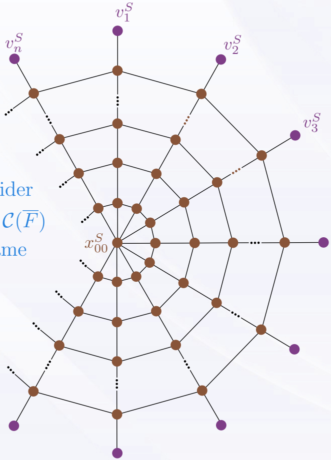

a weight ω_S in $\overline{F}_S$ does not adapt as a weight on the product!

We take σ s. t. $\overline{\omega} = \sigma^{-1}\omega$ on $\overline{F}$ is a weight on the product

Green operator on spider networks

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ω_S weight on $\overline{F}_S$, $\quad \lambda > 0$ real value

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Proposition

Schrödinger operator in terms of a product network

$$\mathcal{L}^S_{q_S}(u_S) = \mathcal{L}_q(u) \qquad \text{on } F_S \setminus \{x^S_{00}\}$$

$$\mathcal{L}^S_{q_S}(u_S)(x^S_{00}) = \left(\lambda + \frac{1}{\overline{\omega}(x_{11})}\sum_{t=1}^{n} c(x_{t1}, x_{t2})\overline{\omega}(x_{t2})\right) u(x_{11}) - \sum_{t=1}^{n} c(x_{t1}, x_{t2}) u(x_{t2})$$

where $q_S = q_{\omega_S} + \lambda$ *and* $q = q_{\overline{\omega}} + \lambda$

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It is given by

$$u_S = \mathcal{G}_q(v\chi_F) + \alpha \sum_{t=1}^{n} c(x_{t1}, x_{t2})\mathcal{G}_q(\varepsilon_{x_{t2}}) + \alpha\chi_{H_1}$$

on $\overline{F}_S$, where α is a constant.

Gracias!