On the Generalization of Dedekind Modules

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Outline

A joint work

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- It has been investigating module theory over commutative domains from the view-point of arithmetic ideal theory (Ali (2006), El-Bast and Smith (1988) , Naoum and Al-Alwan (1996), Saraç, Smith, Tiras (2007)). They mainly focus on multiplication modules except for Dedekind modules.
- However if M is a projective R-module with the uniform dimension n, where R is a Dedekind domain, then M is neither a multiplication module nor a Dedekind module if $n \geq 2$.
- It turns out that M is a generalized Dedekind module.
- We have started studying modules theory over commutative domains, without the condition: M is a multiplication module

Some notions

- R is an integrally closed domain with its quotient field K.
- M is a finitely generated torsion-free R-module with its quotient module KM.
- $R[X]$ is a polynomial ring over R in an indeterminate X
- $M[X]$ is a polynomial $R[X]$-module.
- $K(X)$ is the quotient field of $K[X]$.

Definition 1

An R-submodule N of KM is called a *fractional R-submodule* in KM if there is a $0 \neq r \in R$ such that $rN \subseteq M$ and $KN = KM$. If $M \supseteq N$, then N is a *integral* submodule of M.

Lemma 1

Let N be a fractional R-submodule. Then $\mathfrak{n} = (N : M) = \{r \in R \mid rM \subseteq N\}$ is a non-zero ideal.

For any R-submodule $\mathfrak{a}$ of K, we denote

$$\mathfrak{a}^+ = \{m \in KM \mid \mathfrak{a}m \subseteq M\}.$$

Definition 2

An R-submodule $\mathfrak{a}$ of K is called a *fractional M-ideal* if there is a $0 \neq m \in M$ such that $\mathfrak{a}m \subseteq M$ and $K\mathfrak{a}^+ = KM$. If $R \supseteq \mathfrak{a}$, then $\mathfrak{a}$ is just a non-zero ideal.

Lemma 2

1. Any fractional R-ideal in K is a fractional M-ideal.
2. Let N be an R-submodule of KM. Then $KN = KM$ if and only if N is an essential R-submodule of KM.

For a fractional R-submodule N in KM, we define

- $N^- = \{k \in K \mid kN \subseteq M\}$, a fractional M-ideal in K.
- $N_v = (N^-)^+$ which is a fractional R-submodule in KM and $N_v \supseteq N$.

Definition 3

A fractional R-submodule N in KM is called a *v-submodule* in KM if $N = N_v$.

Lemma 3

Let N be a fractional R-submodule in KM. Then

1. $M = M_v$.
2. $(kN)_v = kN_v$ for any $k \in K$.
3. $N^- = (N_v)^-$.

We recall that M is called a *Dedekind modules* if each submodule N of M is invertible, that is, $N^{-}N = M$.

Definition 4

A module M is called a *generalized Dedekind module* (a G-Dedekind module for short) if

a. each v-submodule of M is invertible and
b. M satisfies the ascending chain condition on v-submodules of M.

Lemma 4

Let R be an integrally closed domain. Let $\mathfrak{a}$ be an invertible fractional ideal in K and let N be a fractional R-submodule of KM. Then $(\mathfrak{a}N)_v = \mathfrak{a}N_v$.

Theorem 1

Suppose R is a Dedekind domain and M is a finitely generated torsion-free R-module. Then

1. Each v-submodule N of M is the form: $N = \mathfrak{n}M$ for some ideal $\mathfrak{n}$ of R and $\mathfrak{n} = (N : M)$.
2. M is a G-Dedekind module.

Sketch of Proof :

First we show that for any P a maximal v-submodule of M (submodules maximal amongst the v-submodules of M), P is a prime submodule of M such that $\mathfrak{p} = (P : M) \neq (0)$ is a prime ideal of R.

Then we show that for any P a prime v-submodule of M, $P = \mathfrak{p}M$, where $\mathfrak{p} = (P : M)$.

Conversely, let $P = \mathfrak{p}M$, where $\mathfrak{p}$ is a maximal ideal of R. Then we show that P is a prime v-submodule of M.

Finally we prove that each v-submodule N of M is of the form $N = \mathfrak{n}M$ for some ideal $\mathfrak{n}$ of R.

Recall Proposition 3.6 and Theorem 3.12 of paper of Alkan, Saraç, Tiras (2005) and Theorem 3.1 of paper of El-Bast and Smith (1988).
We prove it from generalized Dedekind modules point of view.

Corollary 1

Let R be an integrally closed domain with its quotient field K and M a finitely generated torsion-free R-module. If M is a Dedekind module, then $u - \mathrm{dim}\ \mathrm{M} = 1$.

Proposition 1

Let M be a finitely generated torsion-free R-module and R be a Noetherian integrally closed domain.Then M is a Dedekind module if and only if M is a multiplication module with u-dim $M = 1$ and R is a Dedekind domain.

Proposition 2

Let M be a finitely generated torsion-free R-module, where R is an integrally closed domain. Then M is a Noetherian valuation module if and only if M is a multiplication module with u-dim $M = 1$ and and R is a Noetherian valuation domain.

Lemma 5

Let $\mathfrak{n}$ be a fractional R-submodule with $M \supseteq \mathfrak{n}$ and $N = \mathfrak{n}[X]$. Then

1. $N^{-} = \mathfrak{n}^{-}[X]$.
2. $N_v = \mathfrak{n}_v[X]$.

Lemma 6

Let P be a prime $R[X]$- submodule of $M[X]$ with $\mathfrak{p} = P \cap M \neq (0)$. Then

1. $\mathfrak{p}$ is a prime submodule of M.
2. $P_1 = \mathfrak{p}[X]$ is a prime submodule of $M[X]$.

Lemma 7

Let P be a prime v-submodule of $M[X]$ such that $\mathfrak{p} = P \cap M \neq (0)$. Then

(1) $\mathfrak{p}$ is a prime v-submodule of M and $\mathfrak{p} = \mathfrak{p}_0 M$, where $\mathfrak{p}_0$ is a maximal ideal of R with $\mathfrak{p}_0 = (\mathfrak{p} : M)$.

(2) $P = \mathfrak{p}[X] = \mathfrak{p}_0[X]M[X]$, and $\mathfrak{p}_0[X] = (P : M[X])$ is a minimal prime ideal of $R[X]$.

Proposition 3

Let N be a v-submodule of $M[X]$ with $\mathfrak{n} = N \cap M \neq (0)$. Then

(1) $\mathfrak{n}$ is a v-submodule of M and $\mathfrak{n} = \mathfrak{n}_0 M$ for some ideal $\mathfrak{n}_0$ of R.

(2) $N = \mathfrak{n}_0[X]M[X]$ and $\mathfrak{n}_0[X] = (N : M[X])$.

Theorem 2

Let R be a Dedekind domain and M be a finitely generated torsion-free R-module. Then

(1) The $R[X]$-module $M[X]$ is a generalized Dedekind module.

(2) Any v-submodule N of $M[X]$ is of the form: $N = \mathfrak{n}M[X]$, where $\mathfrak{n} = (N : M[X])$.

Let N be a v-submodule of $M[X]$.
If $\mathfrak{n} = N \cap M \neq (0)$, then $N = \mathfrak{n}_0[X]M[X]$. Hence N is an invertible submodule of $M[X]$ since $\mathfrak{n}_0[X]$ is an invertible ideal of $R[X]$.
In case $N \cap M = (0)$, if N is a maximal v-submodule of $M[X]$, then $N = \mathfrak{p}M[X]$ for some minimal prime ideal of $R[X]$, which is invertible. Thus N is an invertible submodule of $M[X]$.
Suppose there is a v-submodule N of $M[X]$ with $N \cap M = (0)$ and N is not invertible. We may assume that N is maximal for this property. Then there is a maximal v-submodule $P = \mathfrak{p}M[X]$ with $P \supset N$, where $\mathfrak{p}$ is a minimal prime ideal of $R[X]$ and $M[X] \supseteq \mathfrak{p}^{-1}N \supseteq N$. If $\mathfrak{p}^{-1}N = N$, then $\mathfrak{p}^{-1} \subseteq R[X]$ by the determinant argument, a contradiction.
If $\mathfrak{p}^{-1}N \cap M \neq (0)$, then $\mathfrak{p}^{-1}N = \mathfrak{m}_0[X]M[X]$ for some invertible ideal $\mathfrak{m}_0[X]$ of $R[X]$ and so $N = \mathfrak{p}\mathfrak{m}_0[X]M[X]$, an invertible submodule of $M[X]$, a contradiction.
If $\mathfrak{p}^{-1}N \cap M = (0)$, then by the choice of N, $\mathfrak{p}^{-1}N$ is invertible and so $M[X] = (\mathfrak{p}^{-1}N)^{-}\mathfrak{p}^{-1}N = \mathfrak{p}N^{-}\mathfrak{p}^{-1}N = N^{-}N$, a contradiction.

Sketch of Proof (2)

We assume that there is a v-submodule N of $M[X]$ such that $N \neq \mathfrak{n}M[X]$, where $\mathfrak{n} = (N : M[X])$. We may assume that N is maximal for this property. Then as in (1), $\mathfrak{p}^{-1}N = \mathfrak{m}M[X]$, where $\mathfrak{m} = (\mathfrak{p}^{-1}N : M[X])$, an invertible ideal of $R[X]$. Thus $N = \mathfrak{p}\mathfrak{m}M[X]$, a contradiction. Hence $N = \mathfrak{n}M[X]$ for all v-submodule N of $R[X]$, where $\mathfrak{n} = (N : M[X])$.

We denote by $F(M)$ the set of all fractional R-submodules in KM. Recall the definition of $*$-operation as follow:

Definition 5

A mapping $N \longrightarrow N^*$ of $F(M)$ into $F(M)$ is called a **-operation* on M if the following conditions hold for each $k \in K$ and all $N, N_1 \in F(M)$:

(i) $(kN)^* = kN^*$.

(ii) $N \subseteq N^*$ and if $N \subseteq N_1$, then $N^* \subseteq N_1^*$.

(iii) $(N^*)^* = N^*$.

Lemma 8

The mapping v: $F(M) \longrightarrow F(M)$ given by $N \longrightarrow N_v$ is a *-operation on M.

Lemma 9

(1) Let N be a fractional R-submodule in KM. Then $N_v = \cap_{N \subseteq kM} kM$, where $k \in K$.

(2) Let $\mathfrak{a}$ be a fractional R- ideal . Then $(\mathfrak{a}M)_v = (\mathfrak{a}_v M)_v$.

(3) Let N be a fractional R-submodule in KM such that $M \supseteq N$ and $N = N_v$ Then $\mathfrak{n} = (N : M) = \{r \in R \mid rm \subseteq N\}$ is a v-ideal of R.

We denote by

- $F(R)$ the set of all fractional v-submodules in KM,
- $F_v(M)$ the Abelian group of fractional ideals in K.

Proposition 4

The mapping : $F(R) \longrightarrow F_v(M)$ given by $\mathfrak{n} \longrightarrow \mathfrak{n}M$ is a bijection, where $\mathfrak{n} \in F(R)$.

Let

- $F_v(M[X]) = \{N \mid N \text{ are fractional } v\text{-submodules in } K(X)M[X]\}$,
- $F_v(R[X]) = \{\mathfrak{n} \mid \mathfrak{n} \text{ are fractional } v\text{-ideals in } K(X)\}$.

Proposition 5

Let R be a commutative Dedekind domain and M be a finitely generated torsion-free R-module. Then the mapping :

$$F_v(R[X]) \longrightarrow F_v(M[X])$$

given by $\mathfrak{n} \longrightarrow \mathfrak{n}M[X]$ is a bijection, where $\mathfrak{n} \in F_v(R[X])$.

- Since $R[X]$ is a generalized Dedekind domain, $F_v(R[X])$ is an Abelian group under the usual ideal product.
- We define a product "$\circ$" in $F_v(M[X])$ as follows:

$$N \circ N_1 = \mathfrak{n}\mathfrak{n}_1 M[X]$$

for $N = \mathfrak{n}M[X]$ and $N_1 = \mathfrak{n}_1 M[X]$, where $\mathfrak{n}, \mathfrak{n}_1 \in F_v(R[X])$.

Corollary 2

$F_v(M[X])$ is isomorphic to $F_v(R[X])$ as Abelian groups.

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THANK YOU