Lattices and cohomological Mackey functors for finite cyclic *p*-groups

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(joint work with Blas Torrecillas)

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Cohomological Mackeyfunctors, I

... in the footsteps of A. Dress

Definition (A. Dress, 1972)

Let G be a finite group, and let **ab** be an abelian category. An object in the category

$$\mathfrak{cMF}_G(\mathbf{ab}) = \mathfrak{Add}^-(\mathbf{perm}_{\mathbb{Z}}(G), \mathbf{ab})$$

is called a **cohomological** G**-Mackey functor with values in the category ab.**

A. Dress

- $\mathbf{perm}_{\mathbb{Z}}(G)$ = additive category of left $\mathbb{Z}[G]$-permutation modules.
- $\mathfrak{Add}^-(\mathscr{C}_1, \mathscr{C}_2)$ = the category of contravariant additive functors from the additive category $\mathscr{C}_1$ to the additive category $\mathscr{C}_2$.
- G.Y.$\Longrightarrow$ $\mathfrak{cMF}_G(\mathbf{ab})$ is an abelian category with enough projectives.

Cohomological Mackeyfunctors, II

... in the footsteps of A. Dress

- $\mathbf{perm}_{\mathbb{Z}}(G)$ is additively generated by $\mathbb{Z}[G/U]$, $U \subseteq G$. Hence $\mathbf{X} \in \mathfrak{cMF}_G(\mathbf{ab})$ is uniquely determined by the values
  - $\mathbf{X}_U = \mathbf{X}(\mathbb{Z}[G/U])$, $U \subseteq G$; and
  - $\mathbf{X}(\phi)$, $\phi \in \mathrm{Hom}_G(\mathbb{Z}[G/U], \mathbb{Z}[G/V])$, $U, V \subseteq G$.

Theorem (A. Dress, 1972)

As a category $\mathbf{perm}(\mathbb{Z}[G])$ *is generated by the morphisms*

- $c_{g,U} \colon \mathbb{Z}[G/U] \to \mathbb{Z}[G/{}^{g}U]$, $g \in G$, $U \subseteq G$, $c_{g,U}(xU) = xg^{-1\,g}U$;
- $\mathfrak{i}_{V,U} \colon \mathbb{Z}[G/V] \to \mathbb{Z}[G/U]$, $U, V \subseteq G$, $V \subseteq U$, $\mathfrak{i}_{V,U}(xV) = xU$;
- $\mathfrak{t}_{V,U} \colon \mathbb{Z}[G/U] \to \mathbb{Z}[G/V]$, $U, V \subseteq G$, $V \subseteq U$, $\mathfrak{t}_{U,V}(xU) = \sum_{r \in \mathscr{R}} xrV$; *where* $\mathscr{R} \subseteq U$ *is a set of representatives for* U/V.

In particular, a cohomological G*-Mackeyfunctor* $\mathbf{X}$ *is uniquely determined by the values*

- $c^{\mathbf{X}}_{g,U} = \mathbf{X}(c_{g,U}) \colon \mathbf{X}_{{}^{g}U} \to \mathbf{X}_U$, $g \in G$, $U \subseteq G$;
- $i^{\mathbf{X}}_{U,V} = \mathbf{X}(\mathfrak{i}_{V,U}) \colon \mathbf{X}_U \to \mathbf{X}_V$, $V \subseteq U$;
- $t^{\mathbf{X}}_{V,U} = \mathbf{X}(\mathfrak{t}_{U,V}) \colon \mathbf{X}_V \to \mathbf{X}_U$, $V \subseteq U$

Cohomological Mackeyfunctors, III

. . . continued

Theorem (continued)

which satisfy the following relations:

(cMF$_1$) $i^{\mathbf{X}}_{U,U} = t^{\mathbf{X}}_{U,U} = c^{\mathbf{X}}_{u,U} = \mathrm{id}_{\mathbf{X}_U}$ *for all* $U \subseteq G$ *and all* $u \in U$*;*

(cMF$_2$) $i^{\mathbf{X}}_{V,W} \circ i^{\mathbf{X}}_{U,V} = i^{\mathbf{X}}_{U,W}$ *and* $t^{\mathbf{X}}_{V,U} \circ t^{\mathbf{X}}_{W,V} = t^{\mathbf{X}}_{W,U}$ *for all* $U, V, W \subseteq G$ *and* $W \subseteq V \subseteq U$*;*

(cMF$_3$) $c^{\mathbf{X}}_{h,{}^gU} \circ c^{\mathbf{X}}_{g,U} = c^{\mathbf{X}}_{hg,U}$ *for all* $U \subseteq G$ *and* $g, h \in G$*;*

(cMF$_4$) $i^{\mathbf{X}}_{{}^gU,{}^gV} \circ c^{\mathbf{X}}_{g,U} = c^{\mathbf{X}}_{g,V} \circ i^{\mathbf{X}}_{U,V}$ *for all* $U, V \subseteq G$ *and* $g \in G$*;*

(cMF$_5$) $t^{\mathbf{X}}_{{}^gV,{}^gU} \circ c^{\mathbf{X}}_{g,V} = c^{\mathbf{X}}_{g,U} \circ t^{\mathbf{X}}_{V,U}$ *for all* $U, V \subseteq G$ *and* $g \in G$*;*

(cMF$_6$) $i^{\mathbf{X}}_{U,W} \circ t^{\mathbf{X}}_{V,U} = \sum_{g \in W\backslash U/V} t^{\mathbf{X}}_{{}^gV\cap W,W} \circ c^{\mathbf{X}}_{g,V\cap W^g} \circ i^{\mathbf{X}}_{V,V\cap W^g}$, *where* $W^g = g^{-1}Wg$ *for all* $U, V, W \subseteq G$ *and* $V, W \subseteq U$*;*

(cMF$_7$) $t^{\mathbf{X}}_{V,U} \circ i^{\mathbf{X}}_{U,V} = |U : V|.\mathrm{id}_{\mathbf{X}_U}$ *for all subgroups* $U, V \subseteq G$*,* $V \subseteq U$*.*

Cohomological Mackeyfunctors - Examples

The fixed point functor

Let G be a finite group, let R be a commutative ring, and let M be a left $R[G]$-module.

- For $U \in G^\sharp$ put $\mathbf{h}^0(M)_U = M^U$.
- For $V \subseteq U$ let $i_{U,V}^{\mathbf{h}^0(M)}\colon M^U \to M^V$ denote the canonical map,
- and for $g \in G$ let $c_{g,U}^{\mathbf{h}^0(M)}\colon M^{{}^g U} \to M^U$ be given by multiplication with $g^{-1} \in G$.
- For $V \subseteq U$ let $\mathscr{R} \subseteq U$ be a set of representatives of U/V, and let $t_{V,U}^{\mathbf{h}^0(M)}\colon M^V \to M^U$ be given by $t_{V,U}^{\mathbf{h}^0(M)}(m) = \sum_{r \in \mathscr{R}} r \cdot m$ for $m \in M^V$.

Then $\mathbf{h}^0(M)$ together with the maps $i_{U,V}^{\mathbf{h}^0(M)}$, $t_{V,U}^{\mathbf{h}^0(M)}$ and $c_{g,U}^{\mathbf{h}^0(M)}$ is a cohomological G-Mackey functor - the **fixed point functor of** M. Thus $\mathbf{h}^0\colon {}_{R[G]}\mathbf{mod} \longrightarrow \mathfrak{cMF}_G({}_R\mathbf{mod})$ is a covariant additive left exact functor. On the contrary, $\_{}_{\{1\}}\colon \mathfrak{cMF}_G({}_R\mathbf{mod}) \longrightarrow {}_{R[G]}\mathbf{mod}$ is an exact functor.

Lattices

... some classical representation theory

Definition

Let G be a finite group, let R be an integral domain, and let $R[G]$ denote the R-group algebra of G. A left $R[G]$-module L which is - considered as R-module - finitely generated and projective is called a left $R[G]$-lattice.

Theorem (B. Torrecillas & T.W. (2013))

Let R be an unramified $(0,p)$ discrete valuation domain, i.e., R is a d.v.d. of characteristic 0 with maximal ideal pR, let G be a finite cyclic p-group, and let L be an $R[G]$-lattice. Then the following are equivalent:

(1) *L is an $R[G]$-permutation module;*

(2) $\mathbf{h}^0(L) \in \mathfrak{cMF}_G({}_R\mathbf{mod})$ *is projective;*

(3) $H^1(L,U) = 0$ *for all $U \subseteq G$ (Hilbert 90 property).*

Remark

The equivalence $(1) \Leftrightarrow (2)$ has been shown already by P. Webb and J. Thévenaz.

Lattices II

What we were not aware off:

Remark

In 1975 S. Endô and T. Miyata proved already that for a finite group G with cyclic p-Sylow subgroups, a $\mathbb{Z}[G]$-lattice L is a direct summand of a $\mathbb{Z}[G]$-permutation lattices, if and only if, $H^1(U, L) = 0$ for all $U \subseteq G$.

The Krull-Schmidt theorem

. . . the ground zero theorem

Theorem (Krull-Schmidt)

Let G be a finite group, and let R be a complete (0,p)-d.v.d. Then, for every left $R[G]$-lattice L the summands L_j of a direct decomposition

$$L = \bigoplus_{1 \le j \le r} L_j$$

into directly indecomposable $R[G]$-lattices L_j are uniquely determined by L.

Theorem (Diederichsen, 1940)

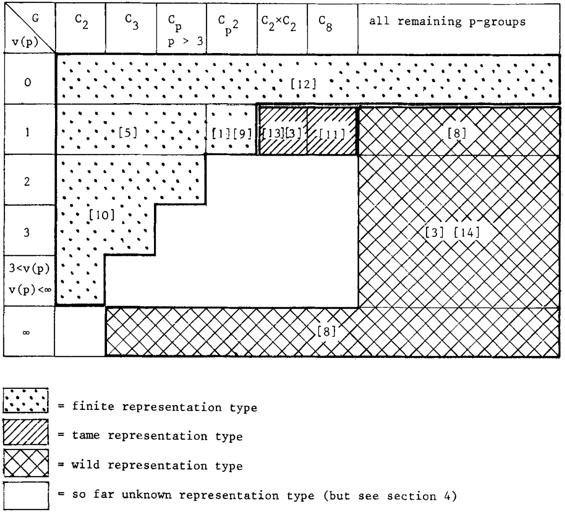
Let G be a cyclic group of order p, let R be an unramified complete d.v.d, and let L be an indecomposable $R[G]$-lattice. Then

$$[L] \in \{\, [R], [R[G]], [\omega_{R[G]}] \,\},$$

where $\omega_{R[G]} = \ker(\varepsilon \colon R[G] \to R)$ is the augmentation ideal of $R[G]$, and $[\_]$ denotes the isomorphism type.

Representation types

. . . from finite to wild (from E. Dieterich (1980))



$C_n = \mathbb{Z}/p\mathbb{Z}$
0: fields of characteristic 0
1: unramified (0,p)-complete d.v.d's
$\infty : R = \mathbb{F}[\![T]\!]$, $char(\mathbb{F}) = p$.

Some implications of our theorem

Wild representation type versus finite presentability by permutation modules

Theorem (B. Torrecillas & T.W., 2013)

Let G be a finite cyclic p-group, and let R be an unramified $(0,p)$-d.v.d. Then

$$\mathrm{gl.dim}(\mathfrak{cMF}_G({}_R\mathbf{mod})) \leq 3.$$

Theorem (B. Torrecillas & T.W., 2013)

Let G be a finite cyclic p-group, and let R be an unramified $(0,p)$-d.v.d. Then for every left $R[G]$-lattice L, there exist left G-sets Ω and Υ and a short exact sequence of left $R[G]$-modules

$$0 \longrightarrow R[\Upsilon] \longrightarrow R[\Omega] \longrightarrow L \longrightarrow 0$$

Comment (A. Zalesskii (2013))

Although this theorem is not in contradiction to anything, it is difficult to accept it. It is in contrast to our intuition.

Section cohomology

or "Hilbert 90 properties" ... (D. Hilbert (1862-1943))

Definition

Let G be a finite group and $\mathbf{X} \in \mathrm{ob}(\mathfrak{cMF}_G({}_{\mathbb{Z}}\mathbf{mod}))$.

- $\mathbf{X}$ is called *i*-**injective** if for all $U, V \in G^\sharp$, $V \subseteq U$, $i^{\mathbf{X}}_{U,V} \colon \mathbf{X}_U \to \mathbf{X}_V$ is injective.
- $\mathbf{X}$ is said to be **of type** H^0 (or of Galois descent) if it is *i*-injective, and for all $U, V \in G^\sharp$, $V \triangleleft U$, the induced map $\tilde{i}^{\mathbf{X}}_{U,V} \colon \mathbf{X}_U \longrightarrow (\mathbf{X}_V)^U$ is an isomorphism.
- $\mathbf{X}$ is said to have the **Hilbert' 90 property**, if it is of type H^0 and for all $U, V \subseteq G$, $V \triangleleft U$, one has $H^1(U/V, \mathbf{X}_V) = 0$.

Remark

$\mathbf{X}$ of type H^0
$\Longleftrightarrow$
$\mathbf{X} \simeq \mathbf{h}^0(\mathbf{X}_{\{1\}})$.

Theorem (D. Hilbert, E. Noether)

Let L/K be a finite Galois extension. Then $(L^\bullet)^\times = \mathbf{h}^0(L^\times)$ is a cohomological $\mathrm{Gal}(L/K)$-Mackey functor with the Hilbert 90 property.

Section cohomology, part II

Definition

Let G be a finite group, $U, V \subseteq G$, $V \triangleleft U$, and let $\mathbf{X} \in \mathrm{ob}(c\mathfrak{M}\mathfrak{F}_G({}_{\mathbb{Z}}\mathbf{mod}))$. Then

$$\mathbf{k}^0(U/V, \mathbf{X}) = \ker(i^{\mathbf{X}}_{U,V}), \qquad \mathbf{c}_0(U/V, \mathbf{X}) = \mathrm{coker}(t^{\mathbf{X}}_{V,U}),$$
$$\mathbf{k}^1(U/V, \mathbf{X}) = \mathbf{X}^U_V/\mathrm{im}(i^{\mathbf{X}}_{U,V}), \quad \mathbf{c}_1(U/V, \mathbf{X}) = \ker(t^{\mathbf{X}}_{V,U})/\omega_{U/V} \cdot \mathbf{X}_V,$$

where $\omega_{U/V} = \ker(\mathbb{Z}[U/V] \to \mathbb{Z})$ is the augmentation ideal, are called the **section cohomology groups** of $\mathbf{X}$ for the normal section (U, V).

Remark

If (U/V) is a cyclic normal section, i.e., U/V is cyclic, then there exists a cohomological U/V-Mackey functor $\mathbf{B}$ such that

$$\mathbf{k}^0(U/V, \mathbf{X}) = \mathrm{Ext}^3(\mathbf{B}, \mathrm{res}_{U/V}(\mathbf{X})), \quad \mathbf{c}_0(U/V, \mathbf{X}) = \mathrm{Ext}^0(\mathbf{B}, \mathrm{res}_{U/V}(\mathbf{X})),$$
$$\mathbf{k}^1(U/V, \mathbf{X}) = \mathrm{Ext}^2(\mathbf{B}, \mathrm{res}_{U/V}(\mathbf{X})), \quad \mathbf{c}_1(U/V, \mathbf{X}) = \mathrm{Ext}^1(\mathbf{B}, \mathrm{res}_{U/V}(\mathbf{X})).$$

Section cohomology, part III

Theorem (T.W. (2006))

Let G be a finite group, $U, V \in G^\sharp$, $V \triangleleft U$, and let $\mathbf{X} \in \mathrm{ob}(c\mathfrak{M}\mathfrak{F}_G({}_{\mathbb{Z}}\mathbf{mod}))$. Then one has a 6-term exact sequence

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbf{c}_1(U/V,\mathbf{X}) & \longrightarrow & \widehat{\mathrm{H}}^{-1}(U/V,\mathbf{X}_V) & \longrightarrow & \mathbf{k}^0(U/V,\mathbf{X}) \\ & & & & & & \downarrow \\ 0 & \longleftarrow & \mathbf{k}^1(U/V,\mathbf{X}) & \longleftarrow & \widehat{\mathrm{H}}^0(U/V,\mathbf{X}_V) & \longleftarrow & \mathbf{c}_0(U/V,\mathbf{X}) \end{array}$$

Remark

If U/V is cyclic and if $\mathbf{X}_W$ are finitely generated abelian groups for all $W \subseteq U$, $V \subseteq W$, then

$$\chi_{U/V}(\mathbf{X}) = \frac{|\mathbf{k}^0(U/V,\mathbf{X})| \cdot |\mathbf{c}_1(U/V,\mathbf{X})|}{|\mathbf{k}^1(U/V,\mathbf{X})| \cdot |\mathbf{c}_0(U/V,\mathbf{X})|} = h(U/V,\mathbf{X}_V)^{-1}$$

coincides with the inverse of the **Herbrand quotient** of $\mathbf{X}_V$.

A Hilbert theorem 90 in a group theoretical context

joint work with Claudio Quadrelli...

Let G be a group and let $N \triangleleft G$ be a subgroup of finite index. For $U \subseteq G/N$, let

$$\widehat{U} = \{ g \in G \mid gN \in U \}$$

and denote by $\widehat{U}^{\mathrm{ab}} = \widehat{U}/[\widehat{U}, \widehat{U}]$ its maximal abelian quotient. Then $\mathbf{X}$ given by $\mathbf{X}_U = \widehat{U}^{\mathrm{ab}}$, where $t^{\mathbf{X}}_{V,U}$ is the canonical map, and $i^{\mathbf{X}}_{U,V}$ is given by the transfer, is a cohomological G/N-Mackey functor (with coefficients in ${}_{\mathbb{Z}}\mathbf{mod}$), we will denote from now on by $\mathbf{h}_1(G/N, \mathbb{Z})$.

C.Quadrelli

Theorem (C.Quadrelli, T.W. (2015))

Let G be a group, and let N be a co-cyclic normal subgroup of finite index, i.e., G/N is a cyclic group. Then

$$\mathbf{c}_1(G/N, \mathbf{h}_1(G/N, \mathbb{Z})) = 0.$$

A Hilbert theorem 90 in a group theoretical context

an immediate consequence . . .

Theorem (C.Quadrelli, T.W. (2015))

Let G be a finitely generated pro-p group, and let N be an open normal co-cyclic subgroup of G, such that U^{ab} is torsion free for every open subgroup U containing N. Then N^{ab} is a $\mathbb{Z}_p[G/N]$-permutation module.

Hilbert's theorem 94

in the classical form ...

Theorem (D. Hilbert, 1897)

Let L/K be a finite Galois extension of number fields, such that

(i) $G = \mathrm{Gal}(L/K)$ *is cyclic,*

(ii) L/K *is unramified.*

Then $|G|$ *divides* $|\ker(\mathrm{Cl}(\mathcal{O}_K) \longrightarrow \mathrm{Cl}(\mathcal{O}_L))|$.

Remark

Hilbert's theorem 94 was the motivation for D. Hilbert to formulate his "principal ideal conjecture" which was proved more than 30 years later by Ph. Furtwängler.

Hilbert's theorem 94 in a stronger form

a kind of "vintage result" ...

Theorem (C.Quadrelli & T.W. (2015))

Let L/K be a finite Galois extension of number fields, such that

(i) $G = \mathrm{Gal}(L/K)$ *is cyclic,*

(ii) L/K *is unramified.*

Then

$$|\ker(\mathrm{Cl}(\mathcal{O}_K) \longrightarrow \mathrm{Cl}(\mathcal{O}_L))| = |G| \cdot |\mathrm{coker}(\mathrm{Cl}(\mathcal{O}_K) \longrightarrow \mathrm{Cl}(\mathcal{O}_L)^G)|.$$

The Schreier index formula

Theorem (O. Schreier)

Let F be a free group of rank $d < \infty$, and let $U \subseteq F$ be a subgroup of finite index. Then U is free of rank

$$\mathrm{rk}(U) = |F : U| \cdot (d - 1) + 1$$

O.Schreier
(1901-1929)

The transfer ratio

Definition

For a finitely generated pro-p group G the non-negative integer

$$\mathrm{rk}_{\mathbb{Q}_p}(G) = \dim_{\mathbb{Q}_p}(G^{\mathrm{ab}} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p)$$

is called the **torsion free rank** of G.

- Let G be a finitely generated pro-p group, and let $U \subseteq G$ be a subgroup of index p. Then U is normal in G, and open.
- Moreover, the transfer map

$$\mathrm{tr}_{G,U} \colon G^{\mathrm{ab}} \longrightarrow U^{\mathrm{ab}}$$

has finite kernel, and $\mathrm{im}(\mathrm{tr}_{G,U}) \subseteq (U^{\mathrm{ab}})^{G/U}$ is of finite index.
- Define the **transfer ratio** $\rho(G, U)$ by

$$\rho(G, U) = \frac{|\ker(\mathrm{tr}_{G,U})|}{|(U^{\mathrm{ab}})^{G/U}/\mathrm{im}(\mathrm{tr}_{G,U})|}.$$

A generalized Schreier formula

... for the torsion free rank in terms of the transfer ratio

Theorem (C. Quadrelli& T.W., 2015)

Let G be a finitely generated pro-p group, and let $U \subseteq G$ be a (closed) subgroup of index p. Then

$$\mathrm{tf}(U) = p \cdot \mathrm{tf}(G) + (1-p)(1 - \log_p(\rho(G, U))),$$

where $\log_p(\_)$ *denotes the logarithm to the base p.*

Blocks

joint work with B. Lancellotti & S. Koshitani

Let $(\mathcal{O}, \mathbb{F}, \mathbb{K})$ be a quasi-split p-modular system for G, i.e.,

- $\mathcal{O}$ is an unramified (0,p)-complete d.v.d. with residue field $\mathbb{F}$ and quotient field $\mathbb{K}$;
- every Wedderburn component of $\mathbb{F}[G]/\mathrm{rad}(\mathbb{F}[G])$ is an $\mathbb{F}$-matrix algebra;
- every Wedderburn component of $\mathbb{K}[G]$ is an $\mathbb{L}$-matrix algebra, where $\mathbb{L}$ is a finite extension field of $\mathbb{K}$ (totally ramified in the place associated to $\mathcal{O}$).

Definition

Let $(\mathcal{O}, \mathbb{F}, \mathbb{K})$ be a quasi split p-modular system for the finite group G. An indecomposable summand B of the $\mathcal{O}[G]$-bimodule $\mathcal{O}[G]$ is called an $\mathcal{O}[G]$-block, i.e., there exists a central primitive idempotent e_B in B such that $B = \mathcal{O}[G] \cdot e_B$.

Green correspondence

. . .

Remark

Let M be an indecomposable $\mathcal{O}[G]$-lattice. A p-subgroup U for which there exists an $\mathcal{O}[U]$-module S such that M is a direct summand of $\mathrm{ind}_U^G(S)$, but not a direct summand of $\mathrm{ind}_V^G(T)$ for any proper subgroup V of U, is called a **vertex** of M, i.e., ${}^G U = \mathrm{vt}(M)$. The $\mathcal{O}[U]$-module S is called a source of M.

Remark

Let B be an $\mathcal{O}[G]$-block. There exists a p-group D unique up to G-conjugacy such that

$$\Delta(D) = \{ (g, g) \in G \times G \mid g \in D \} = \mathrm{vt}_{G\times G}(B).$$

The group D is called the **defect group** of G.

Alperin's weight conjecture

... the Block form

- A left $\mathcal{O}[G]$-module (or $\mathbb{F}[G]$-module or $\mathbb{K}[G]$-module) M is said to be contained in the $\mathcal{O}[G]$-block $B = e_B \cdot \mathbb{K}[G]$, if $e_B \cdot M = M$.
- $\mathrm{IBr}(B) = \{ [S] \mid S \text{ simple } \mathbb{F}[G]\text{-module in } B \}$.

$$\mathrm{Alp}(G) = \{ (P, [S]) \mid P \subseteq G \text{ a } p\text{-subgroup}, \; S \text{ projective and irreducible } \mathbb{F}[N_G(P)/P]\text{-module.} \}$$

J.L. Alperin

- Elements in $\mathrm{Alp}(G)$ are called **weights**.
- By Green correspondence, every weight $(P, [S])$ determines an indecomposable trivial source $\mathcal{O}[G]$-lattice $T(P, [S])$.
- $\mathrm{Alp}(B) = \{ (P, [S]) \in \mathrm{Alp}(G) \mid T(P, [S]) \in B \}$.

Conjecture (Alperin's weight conjecture, 1990)

For every Block B of a finite group G one has

$$|\mathrm{IBr}(B)| = |\mathrm{Alp}(B)|.$$

Blocks with cyclic defect

. . . by Brauer, Thompson, Dade, et al.

- Using the Brauer tree one sees easily that for every $[S] \in \mathrm{IBr}(B)$ there exist B-lattices $L^{\pm}([S])$.
- There exist trivial source lattices $P_0([S])$ and $P_1([S])$ such that

$$0 \longrightarrow \mathbf{h}^0(P_1([S])) \longrightarrow \mathbf{h}^0(P_0([S])) \longrightarrow \mathbf{h}^0(L^+([S])) \longrightarrow 0$$

is exact. Put $\mathfrak{p}([S]) = [P_0([S])] - [P_1([S])] \in \mathbf{Ts}(B)$, where $\mathbf{Ts}(B)$ denotes the Grothendieck group of trivial source $\mathcal{O}[G]$-modules in B.

- Put $\mathfrak{p}\colon \mathrm{IBr}(B) \longrightarrow \mathbf{Ts}(B)$, $\mathfrak{p}([S]) = [P_0([S])] - [P_1([S])]$.
- Then $\mathfrak{p}([S]) = \sum_{T \in \mathrm{ITs}(B)} a_T \cdot [T]$. Put

$$\mathrm{supp}([S]) = \{\, [T] \in \mathrm{ITs}(B) \mid a_T \neq 0 \,\}$$
$$\mathrm{supp}^{\mathrm{mx}}([S]) = \{\, [T] \in \mathrm{ITs}^{mx}(B) \mid a_T \neq 0 \,\}$$

where $\mathrm{ITs}^{\mathrm{mx}}(B) = \{[T] \in \mathrm{ITs}(B) \mid \mathrm{vt}(T) = \mathrm{df}(B)\,\}$.

Alperin's weight conjeture for Blocks with cyclic defect

Remark

By the general theory of Blocks with cyclic defect, it is well known that Alperin's weight conjecture is true for Blocks with cyclic defect.

Theorem (B. Lancellotti, S. Koshitani, T.W., (2018))

Let B be a $\mathcal{O}[G]$-block of cyclic defect. Then for all $S \in \mathrm{IBr}(B)$, $|\mathrm{supp}^{\mathrm{mx}}([S])| = 1$.

Conjecture (B. Lancellotti, S. Koshitani, T.W.)

(a)The map $\alpha\colon \mathrm{IBr}(B) \longrightarrow \mathrm{ITs}(B)$ given by $\mathrm{supp}^{\mathrm{mx}}([S]) = \{\,[\alpha([S])]\,\}$ is a bijection. In particular, there exists a canonical bijection

$$\alpha\colon \mathrm{IBr}(B) \longrightarrow \mathrm{Alp}(B).$$

(b) For $T \in \mathrm{supp}^{mx}([S])$, $a_T \in \{\pm 1\}$.

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References II

Happy Birthday Blas!