Mixed perverse sheaves on flag varieties of Coxeter groups

Cristian Vay
UNC–CONICET
Argentina

joint work with P. Achar and S. Riche

Almería 2019
On the occasion of Blas 60th birthday

Coxeter system

A finite set S and $\{m_{st}\}_{s,t\in S} \in \mathbb{N} \cup \{\infty\}$ such that

$$m_{ss} = 1 \text{ and } m_{st} = m_{ts} \text{ if } s \neq t.$$

Coxeter group

$$W = \langle s \in S \mid (st)^{m_{st}} = 1 \quad \forall\, s, t \in S \rangle$$

Examples

- The symmetric group $\mathbb{S}_n$ with $S = \{(i\, i+1) \mid 1 \leq i < n\}$
- Weyl group of a finite-dimension semisimple Lie algebra
- Weyl group of an affine Lie algebra

Hecke algebra

$$\mathcal{H} = \mathbb{Z}[v^{\pm 1}]\langle H_s, s \in S \mid \text{ with the following relations } \rangle$$

$$H_s^2 = (v^{-1} - v)H_s + 1 \quad \text{and} \quad \underbrace{H_sH_tH_s\cdots}_{m_{st}} = \underbrace{H_tH_sH_t\cdots}_{m_{ts}}$$

$\forall\, s, t \in S$ with $s \neq t$.

- Let $\{H_w\}_{w \in W}$ be the standard basis of $\mathcal{H}$,

$$H_w = H_{s_1} \cdots H_{s_n}$$

  for any reduced expression of $w = s_1 \cdots s_n \in W$.
- Let $\overline{(\quad)} : \mathcal{H} \to \mathcal{H}$, be the $\mathbb{Z}$-algebra involution induced by

$$v \mapsto v^{-1} \quad \text{and} \quad H_s \mapsto H_s^{-1}$$

Kazhdan, Lusztig. Representations of Coxeter groups and Hecke algebras, Invent. Math. (1979).

Theorem

There exists a unique basis $\{\underline{H}_w\}_{w\in W}$ of $\mathcal{H}$ such taht

$$\overline{\underline{H}_w} = \underline{H}_w \quad \text{and} \quad \underline{H}_w = H_w + \sum_{x<w} h_{x,w} H_x,$$

with $h_{x,w} \in v\mathbb{Z}[v]$.

Conjectures (actually theorems)

- The coefficients of $h_{x,w}$ are positives [Kazhdan-Lusztig for Weyl finite and afines groups].
- $\operatorname{ch} L_w = \sum_{x\leq w} (-1)^{\ell(x)+\ell(w)} h_{x,w}(1) \operatorname{ch} M_x$, for a semisimple complex Lie algebra [Beilinson-Bernstein and Brylinsky-Kashiwara].

$\mathcal{P}$: the category of perverse sheaves

$\mathcal{P}$ is the heart of a t-structura on $\mathcal{D}^{\mathrm{b}}(G/B, \mathbb{C})$

the bounded derived category of B-equivariant complexes (with complex coefficients) on the flag variety G/B of a Kac-Moody

The simple objects are $\mathbf{IC}_w$, w in the Weyl group of G,

the intersection cohomology complexes on the Schubert variety $\overline{BwB/B}$.

Categorification of the Hecke algebra

$$[\mathcal{P}] \xrightarrow{\sim} \mathcal{H}$$

$$[\mathbf{IC}_w] \longmapsto \underline{H}_w$$

- $R = S(\mathfrak{h}^*)$ with $\mathfrak{h}$ the Lie algebra of the maximal torus of G.
- $R^s =$ the s-invariant subalgebra of R.

$\mathbb{S}\mathrm{Bim}$: Soergel Bimodules

is the essential image of the hypercohomology

$$\mathbb{H}^\bullet : \mathcal{P} \longrightarrow R\text{-}\mathrm{Bim},$$

which is a fully faithful monoidal functor.

Example

$$\mathbb{H}^\bullet(\mathbf{IC}_e) \simeq R \quad \text{y} \quad \mathbb{H}^\bullet(\mathbf{IC}_s) \simeq R \otimes_{R^s} R(1) =: B_s.$$

- Let $\mathbb{S}\mathrm{Bim}$ be the idempotent completion of the monoidal subcategory generated by B_s, $s \in S$.

Algebraic categorification of the Hecke algebra

$$\begin{aligned} [\mathbb{S}\mathrm{Bim}] &\xrightarrow{\sim} \mathcal{H} \\ [B_s] &\longmapsto \underline{H}_s \end{aligned}$$

- The indecomposable objects of $\mathbb{S}\mathrm{Bim}$ are parametrized by W.

Soergel Conjeture (actually theorem [Elias-Williamson])

Let B_w be the indecomposable object attached to $w \in W$, then

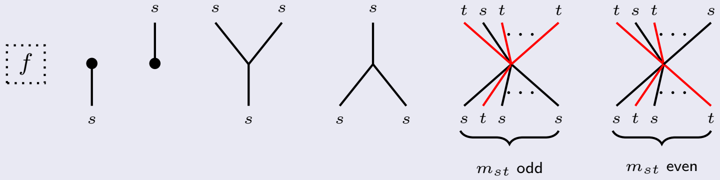
$$[B_w] = \underline{H}_w$$

[Soergel for Weyl and dihedral groups; Fiebig-Libedinsky universal Coxeter groups]

Let $\mathfrak{h}$ be a realization of (W, S) and $R = S(\mathfrak{h}^*)$ with $\operatorname{gr} \mathfrak{h}^* = 2$.

Elias-Williamson diagrammatic categories $\mathscr{D}_{\mathrm{BS}}(\mathfrak{h}, W)$

- Objects: $B_{\underline{w}}$, for any word $\underline{w}$ in S.
- Morphisms: $\Bbbk$-graded modules generated by

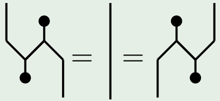


for any $f \in R$ and $s, t \in S$, subject to certain relations

- Tensor product: $B_{\underline{v}} \star B_{\underline{w}} = B_{\underline{vw}}$.

$\Bbbk$ is a Noetherian integral domain of finite global dimension s. t. finitely generated proyective modules are free (for instance, $\Bbbk = \mathbb{Z}$).

Example of a relation



. This implies that B_s is self-dual.

Also, it holds that $B_s \star B_s \cong B_s(1) \oplus B_s(-1)$.

Definition

- $\mathbb{D}$ denotes the autoequivalence in $\mathscr{D}_{\mathrm{BS}}(\mathfrak{h}, W)$ given by flipping diagrams upside-down.
- (1) denotes the shift of grading.

Elias, Williamson. Soergel Calculus. Represent. Theory (2016).

Theorem [EW]

Assume that $\Bbbk$ is a field or a complete local ring and let $\mathscr{D}(\mathfrak{h}, W)$ be the idempotent completion of $\mathscr{D}_{BS}(\mathfrak{h}, W)$. Then

1. The indescomposable objects are parametrized by W.
2. The assignment $[B_s] \mapsto \underline{H}_s$ induces an isomorphism $[\mathscr{D}(\mathfrak{h}, W)] \to \mathcal{H}$ of $\mathbb{Z}[v^{\pm 1}]$-algebras.
3. The Soergel conjecture $[B_w] = \underline{H}_w$ holds for $\Bbbk = \mathbb{R}$.

- $I \subset W$ closed by the Bruhat order,

$$\rightsquigarrow \mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W) = \langle B_{\underline{w}} \mid \ \underline{w} \in I \text{ reduced word}\rangle.$$

- $I = I_0 \setminus I_1$ locally closed, i.e. I_0 and I_1 closed,

$$\rightsquigarrow \mathscr{D}_{\mathrm{BS},I}(\mathfrak{h}, W) = \mathscr{D}^{\oplus}_{\mathrm{BS},I_0}(\mathfrak{h}, W) /\!\!/ \mathscr{D}^{\oplus}_{\mathrm{BS},I_1}(\mathfrak{h}, W)$$

Example

$\mathscr{D}_{\mathrm{BS},\{w\}}(\mathfrak{h}, W) \cong \mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(R)$ but $\mathscr{D}_{\mathrm{BS},W}(\mathfrak{h}, W) \not\cong \mathscr{D}_{\mathrm{BS}}(\mathfrak{h}, W)$

- $I \subset W$ locally closed subset

Definition

$$\mathsf{BE}_I(\mathfrak{h}, W) = K^{\mathrm{b}} \mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h}, W)$$

Example

$\mathsf{BE}_{\{w\}}(\mathfrak{h}, W) \cong D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(R)$ and $\mathsf{BE}_W(\mathfrak{h}, W) \cong K^{\mathrm{b}} \mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)$

Recollement or Gluing

Theorem

J closed finite $\subset I$ locally closed. Then there exists a recollement diagram

$$\begin{tikzcd} \mathsf{BE}_J(\mathfrak{h},W) \arrow[r, "(i^I_J)_*"] & \mathsf{BE}_I(\mathfrak{h},W) \arrow[l, bend right, "(i^I_J)^*"'] \arrow[l, bend left, "(i^I_J)^!"] \arrow[r, "(i^I_{I\setminus J})^*"] & \mathsf{BE}_{I\setminus J}(\mathfrak{h},W). \arrow[l, bend right, "(i^I_{I\setminus J})_!"'] \arrow[l, bend left, "(i^I_{I\setminus J})_*"] \end{tikzcd}$$

and $\mathbb{D}$ interchanges $*$ and $!$.

Among other things, $\forall \mathscr{F} \in \mathsf{BE}_I(\mathfrak{h},W)$ there exist distinguished triangles

$$(i^I_{I\setminus J})_!(i^I_{I\setminus J})^*\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow (i^I_J)_*(i^I_J)^*\mathscr{F} \xrightarrow{+1}$$

$$(i^I_J)_*(i^I_J)^!\mathscr{F} \longrightarrow \mathscr{F} \longrightarrow (i^I_{I\setminus J})_*(i^I_{I\setminus J})^*\mathscr{F} \xrightarrow{+1}$$

Beĭlinson, Bernstein, Deligne. Faisceaux pervers, Astérisque (1982).

Example: the singleton case

Let $w \in I$ minimal, i.e. $\{w\}$ is closed in I, and $x \in I \setminus \{w\}$.

$$
\begin{tikzcd}
\mathsf{BE}_I(\mathfrak{h}, W) & & \mathsf{BE}_{I \setminus \{w\}}(\mathfrak{h}, W) \arrow[ll, bend left, "(i^I_{I \setminus \{w\}})_*"]
\end{tikzcd}
$$

$$
\begin{aligned}
&(i^I_{I \setminus \{w\}})_* B_{\underline{x}} = \\
&\qquad \cdots \to 0 \to B_{\underline{w}} \,\underline{\otimes}_R\, \mathrm{Hom}^\bullet_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{x}}) \xrightarrow{f} B_{\underline{x}} \to 0 \to \cdots ,
\end{aligned}
$$

This is the cone of f and we have a distinguished triangle

$$
B_{\underline{x}} \to (i^I_{I \setminus \{w\}})_* B_{\underline{x}} \to B_{\underline{w}} \,\underline{\otimes}_R\, \mathrm{Hom}^\bullet_{\mathscr{D}^{\oplus}_{\mathrm{BS},I}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{x}})[1] \xrightarrow{[1]}
$$

Example: the singleton case

Let $w \in W$ and $s \in S$ such that $ws > w$. Then

$$\mathrm{Hom}^\bullet_{\mathscr{D}^{\oplus}_{\mathrm{BS},\{w,ws\}}(\mathfrak{h},W)}(B_{\underline{w}}, B_{\underline{ws}}) = R\,\langle\, \mathrm{id}_{B_{\underline{w}}} \star \;\; \rangle$$

and therefore

$$B_{\underline{ws}} \to \left(i_{\{ws\}}^{\{w,ws\}}\right)_* B_{\underline{ws}} \to B_{\underline{w}}\langle 1\rangle \xrightarrow{[1]}$$

$$B_{\underline{w}}\langle -1\rangle \to \left(i_{\{ws\}}^{\{w,ws\}}\right)_! B_{\underline{ws}} \to B_{\underline{ws}} \xrightarrow{[1]}$$

are distinguished triangles in $\mathsf{BE}_{\{w,ws\}}(\mathfrak{h}, W)$.

t-structure

Definition

The *perverse t-structure* in $\mathsf{BE}_I(\mathfrak{h}, W)$ is defined by

$$^p\mathsf{BE}_I(\mathfrak{h}, W)^{\leq 0} = \{\mathscr{F} \mid \forall w \in I,\ (i_w^I)^*(\mathscr{F}) \in {}^p\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)^{\leq 0}\},$$

$$^p\mathsf{BE}_I(\mathfrak{h}, W)^{\geq 0} = \{\mathscr{F} \mid \forall w \in I,\ (i_w^I)^!(\mathscr{F}) \in {}^p\mathsf{BE}_{\{w\}}(\mathfrak{h}, W)^{\geq 0}\}.$$

Definition

The category of *perverse objects* is

$$\mathsf{P}_I^{\mathsf{BE}}(\mathfrak{h}, W) = {}^p\mathsf{BE}_I(\mathfrak{h}, W)^{\leq 0} \cap {}^p\mathsf{BE}_I(\mathfrak{h}, W)^{\geq 0},$$

the heart of the t-structure.

Standard and costandard objects

- b_w is the canonical object in $\mathscr{D}^{\oplus}_{\mathrm{BS},\{w\}}(\mathfrak{h}, W) \cong \mathrm{Free}^{\mathrm{fg},\mathbb{Z}}(R)$.

Definition

$$\Delta^I_w = (i^I_w)_! b_w \quad \text{and} \quad \nabla^I_w = (i^I_w)_* b_w .$$

1. $\Delta^I_w = \nabla^I_w = B_{\underline{w}}$ if $w \in I$ is minimal.
2. $\Delta^I_e = \nabla^I_e = B_{\varnothing}$ if $e \in I$.
3. $\Delta^{\{e,s\}}_s = \cdots 0 \to B_s \longrightarrow B_{\varnothing}(1) \to 0 \cdots$
4. $\nabla^{\{e,s\}}_s = \cdots 0 \to B_{\varnothing}(-1) \longrightarrow B_s \to 0 \cdots$
5. $\mathbb{D}(\Delta^I_w) = \nabla^I_w$.
6. $\Delta^J_w = \Delta^I_w$ if $J \subset I$.

Main results

If $w = s_1 \cdots s_r$ is a reduced expression, then

$$\Delta_w \cong \Delta_{s_1} \underline{\star} \Delta_{s_2} \underline{\star} \cdots \underline{\star} \Delta_{s_r}, \qquad \nabla_w \cong \nabla_{s_1} \underline{\star} \nabla_{s_2} \underline{\star} \cdots \underline{\star} \nabla_{s_r}$$

$$\Delta_w \underline{\star} \nabla_{w^{-1}} \cong \nabla_{w^{-1}} \underline{\star} \Delta_w \cong B_\varnothing$$

Δ_w^I and ∇_w^I are perverse.

The assignment $[B_\varnothing(1)] \mapsto v$ and $[\Delta_w] \mapsto H_w$, $w \in W$, induces an $\mathbb{Z}[v^{\pm 1}]$-algebra isomorphism

$$[\mathsf{BE}_W(\mathfrak{h}, W)] \xrightarrow{\sim} \mathcal{H}.$$

Consequence: $[\mathscr{D}^{\oplus}_{\mathrm{BS}}(\mathfrak{h}, W)] \simeq [\mathsf{BE}_W(\mathfrak{h}, W)] \simeq \mathcal{H}$.

Main results

$$\mathrm{Hom}_{\mathsf{BE}_I(\mathfrak{h},W)}(\Delta^I_x, \nabla^I_y\langle n\rangle[m]) \cong \begin{cases} R^m & \text{if } x = y,\ m = -n \in 2\mathbb{Z}_{\geq 0} \\ 0 & \text{otherwise} \end{cases}$$

Remark

If $\Bbbk$ is a field, the simple perverse objects are given by the recollement. More precisely, let

$$\mathscr{L}^I_w := \mathrm{im}(\Delta^I_w \to \nabla^I_w).$$

Then

- $\mathsf{P}^{\mathsf{BE}}_I(\mathfrak{h}, W)$ is a Jordan-Holder category.
- The simple objects are $\mathscr{L}^I_w\langle n\rangle$ for all $w \in I$ and $n \in \mathbb{Z}$.

Main results

The socle of Δ_w is $\mathscr{L}_e\langle -\ell(w)\rangle$ and the cokernel of $\mathscr{L}_e\langle -\ell(w)\rangle \hookrightarrow \Delta_w$ has no composition factors of the form $\mathscr{L}_e\langle n\rangle$.

The head of ∇_w is $\mathscr{L}_e\langle \ell(w)\rangle$ and the kernel of $\nabla_w \twoheadrightarrow \mathscr{L}_e\langle \ell(w)\rangle$ has no composition factors of the form $\mathscr{L}_e\langle n\rangle$.

For all $w, y \in W$, it holds that

$$\dim \operatorname{Hom}_{\mathsf{BE}(\mathfrak{h},W)}(\Delta_w, \Delta_y\langle n\rangle) = \begin{cases} 1 & \text{if } w \leq y,\ n = \ell(y) - \ell(w) \\ 0 & \text{otherwise} \end{cases}$$

If $w \leq y$, the morphism $\Delta_w \to \Delta_y\langle \ell(y) - \ell(w)\rangle$ is inyective.

Right-equivariant categories

Definition

$$\mathsf{RE}_I(\mathfrak{h}, W) = K^{\mathrm{b}}\overline{\mathscr{D}}_{\mathrm{BS}}^{\oplus}(\mathfrak{h}, W)$$

where $\overline{\mathscr{D}}_{\mathrm{BS}}(\mathfrak{h}, W)$ is the category with the same objects that $\mathscr{D}_{\mathrm{BS}}(\mathfrak{h}, W)$ but Hom-spaces $\Bbbk \otimes_R \mathrm{Hom}^{\bullet}_{\mathscr{D}_{\mathrm{BS}}(\mathfrak{h},W)}$.

Example

$\mathsf{RE}_{\{w\}}(\mathfrak{h}, W) \cong D^{\mathrm{b}}\mathrm{Mod}^{\mathrm{fg},\mathbb{Z}}(\Bbbk)$ and $\mathsf{RE}_W(\mathfrak{h}, W) \cong K^{\mathrm{b}}\mathscr{D}_{\mathrm{BS}}^{\oplus}(\mathfrak{h}, W)$

$\overline{\mathscr{D}}_{\mathrm{BS}}^{\oplus}(\mathfrak{h}, W)$ and $\mathsf{RE}(\mathfrak{h}, W)$ are right modules categories over $\mathscr{D}_{\mathrm{BS}}^{\oplus}(\mathfrak{h}, W)$ and $\mathsf{BE}(\mathfrak{h}, W)$, respectively.

As for the categories $\mathsf{BE}_I(\mathfrak{h}, W)$, we can endow $\mathsf{RE}_I(\mathfrak{h}, W)$ with recollement structures t-structures.

Definition

The heart of $\mathsf{P}^{\mathsf{RE}}(\mathfrak{h}, W)$ of the t-structure of $\mathsf{RE}(\mathfrak{h}, W)$ is *the category of perverse sheaves on* $(\mathfrak{h}, W)$.

Lemma

The forgetful functor

$$\mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}} : \mathsf{BE}_I(\mathfrak{h}, W) \to \mathsf{RE}_I(\mathfrak{h}, W)$$

is t-exact.

Standard and costandard objects

Definition

$$\overline{\Delta}^I_w := \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\Delta^I_w) \quad \text{and} \quad \overline{\nabla}^I_w := \mathsf{For}^{\mathsf{BE}}_{\mathsf{RE}}(\nabla^I_w).$$

Therefore $\overline{\Delta}^I_w$ and $\overline{\nabla}^I_w$ are perverse and it holds that

$$\mathrm{Hom}_{\mathsf{RE}_I(\mathfrak{h},W)}(\overline{\Delta}^I_x, \overline{\nabla}^I_y\langle n\rangle[m]) \cong \begin{cases} \Bbbk & \text{if } x = y \text{ and } m = n = 0 \\ 0 & \text{otherwise} \end{cases}$$

Properties

- Assume that $\Bbbk$ is a field.
- $\overline{\mathscr{L}}_w^I := \mathrm{im}(\overline{\Delta}_w^I \to \overline{\nabla}_w^I)$.

Theorem

$\mathsf{P}_I^{\mathsf{RE}}(\mathfrak{h}, W)$ is a graded highest-weight with standard and costandard objects $\overline{\Delta}_w^I$ and $\overline{\nabla}_w^I$ for all $w \in I$.
The simples objects are $\overline{\mathscr{L}}_w^I$ for all $w \in I$.

Definition

$\mathsf{Tilt}_I^{\mathsf{RE}}(\mathfrak{h}, W)$ is the subcategory of tilting objects (those admiting an standard and a costandard filtration).

The followings hold in any highest weight category, see for instance

Achar, Riche. Modular perverse sheaves on flag varieties II: Koszul duality and formality. Duke Math. J. (2016),

The tilting objects are parametrized by I and the tilting $\mathscr{T}_w^I$, $w \in I$, is characterized by

$$(\mathscr{T}_w^I : \overline{\Delta}_w^I) = 1 \quad \text{and} \quad (\mathscr{T}_w^I : \overline{\Delta}_x^I\langle n\rangle) \neq 0 \Rightarrow x \leq w.$$

There exist equivalences of trinagulated categories

$$K^{\mathrm{b}}\mathsf{Tilt}_I^{\mathsf{RE}}(\mathfrak{h}, W) \to D^{\mathrm{b}}\mathsf{P}_I^{\mathsf{RE}}(\mathfrak{h}, W) \to \mathsf{RE}_I(\mathfrak{h}, W)$$

Ringel duality

- Assume that W is finite and $\Bbbk$ is a field.
- $w_0 \in W$ is the longest element.

$(-)\underline{\star}\Delta_{w_0} : \mathrm{RE}(\mathfrak{h}, W) \to \mathrm{RE}(\mathfrak{h}, W)$ is an equivalence of triangulated categories. Moreover,

$$\overline{\nabla}_x \underline{\star} \Delta_{w_0} \simeq \overline{\Delta}_{xw_0}$$

$$\mathscr{T}_x \underline{\star} \Delta_{w_0} \cong \mathscr{P}_{xw_0}, \qquad \mathscr{I}_x \underline{\star} \Delta_{w_0} \cong \mathscr{T}_{xw_0}.$$

$$\mathscr{T}_{w_0} \cong \mathscr{P}_e\langle \ell(w_0)\rangle \cong \mathscr{I}_e\langle -\ell(w_0)\rangle.$$

$$\left(\mathscr{T}_{w_0} : \overline{\nabla}_x\langle -n\rangle\right) = \left(\mathscr{T}_{w_0} : \overline{\Delta}_x\langle n\rangle\right) = \begin{cases} 1 & \text{if } n = \ell(xw_0); \\ 0 & \text{otherwise} \end{cases}$$

Gracias!

Theorem [EW]

$\mathrm{Hom}^{\bullet}_{\mathscr{D}_{\mathrm{BS}}(\mathfrak{h},W)}(B_{\underline{v}}, B_{\underline{w}})$ is a free R-module of finite rank $\forall\, \underline{v},\ \underline{w}$.

- The double leaves basis is parametrized by

$$\bigcup_{x \in W} M(\underline{w}, x) \times M(\underline{v}, x)$$

and the elements satisfy

$$\begin{tikzcd} B_{\underline{v}} \arrow[rr, "\mathbb{LL}^{w}_{\mathbf{f},\mathbf{e}}"] \arrow[dr, "LL_{\underline{v},\mathbf{e}}"'] & & B_{\underline{w}} \\ & B_{\underline{x}} \arrow[ur, "\overline{LL_{\underline{w},\mathbf{f}}}"'] & \end{tikzcd}$$

Example

 forms a basis of $\mathrm{Hom}^{\bullet}_{\mathscr{D}_{\mathrm{BS}}(\mathfrak{h},W)}(B_s, B_{\varnothing})$.