ON CLEAN COMODULES AND CLEAN COALGEBRAS

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Clean Rings and Clean Modules

Let R be a ring with multiplicative identity.

Definition

(Nicholson, 1977)
A Ring R is called a clean ring if every element in ring R can be expressed as the sum of an idempotent element and a unit in R.

Definition

(Camillo, et.al., 2006)
An R-module M is called a clean module if the ring of $End_R(M)$ is clean.

Comodule as Modules over Dual Algebra

Theorem

(Brzeziński and Wisbauer, 2003)
If (C, Δ, ε) is an R-coalgebra, then C^* is a dual R-algebra over a convolution product $f * g = \mu \circ (f \otimes g) \circ \Delta$, for any $f, g \in C^*$.

Theorem

(Brzeziński and Wisbauer, 2003)
Every right C-comodule M is a left C^*-module by

$$\rightharpoonup : C^* \otimes_R M \to M, f \otimes m \mapsto (I_M \otimes f) \circ \varrho^M(m) = \Sigma m_{\underline{0}} f(m_{\underline{1}}).$$

Remark

Since any R-coalgebra C is a (C, C)-comodule, C is a left and right C^*-module.

Theorem

(Brzeziński and Wisbauer, 2003)
Any morphism $h : M \to N$ *in* $\mathbf{M}^C$ *is a left* C^**-module morphism, that is* $Hom^C(M,N) \subseteq_{C^*} Hom(M,N)$.

Remark

If C satisfies the α-condition, then $Hom^C(M,N) =_{C^*} Hom(M,N)$.

Clean Comodules and Clean Coalgebras

As the fact that a C-comodule M is a C^*-module. we define the notions of clean comodules.

Definition

Let C be an R-coalgebra. A C-comodule M is called a clean comodule **if M is clean as a C^*-module**.

It means C-comodule M is a clean comodule if ${}_{C^*}End(M)$ is clean. Since every R-coalgebra C is a comodule over itself definition of clean coalgebra given as below

Definition

An R-coalgebra C is a clean coalgebra if C is a clean comodule over itself or the ring ${}_{C^*}End(C)$ is clean.

Trivial Clean Coalgebras and Clean Comodules

Every ring R with multiplicative identity is a (trivial) R-coalgebra with $\Delta_T(r) = r \otimes r$ and $\varepsilon_T(r) = 1$, for any $r \in R$. Hence, the dual algebra $R^* = Hom_R(R, R) \simeq R$.

Example

A ring R is a clean ring if and only if $(R, \Delta_T, \varepsilon_T)$ is a clean (trivial) R-coalgebra.

Example

An R-module M is a clean module if and only if M is a clean (trivial) R-comodule with $\varrho^M(m) = m \otimes 1$, for any $m \in M$.

Recall the theorem on clean modules

Theorem

1. **(Camillo et.al, 2006)** *Every continuous modules is clean.*
2. **(Camillo et.al, 2006)** *If M is a quasi-injective R-module, then M is a clean module (every injective module is clean).*
3. **(Nicholson, et.al, 2004)** *Let D be a division ring. Every vector space V over D is a clean ring*

Direct Consequences From Module Theory

Proposition

Let (C, Δ, ε) be an R-coalgebra with the α-conditon.

1. If C^* is a clean ring, then C is a clean R-coalgebra;
2. if C^* is a division ring, then C is a clean R-coalgebra.

Proof. Since C satisfies the α-condition, ${}_{C^*}End(C) \simeq C^*$ (see Brzeziński and Wisbauer (2003)). Then the proof is clear.

Proposition

If C is an R-coalgebra and finitely generated projective R-module, then every injective C-comodule is a clean comodule.

Proposition

Let R be a QF ring, M, C is a f.g R-module. If M is an injective C-comodule, then M is a clean C-comdules.

Clean $R[G]$-comodules

Theorem

(Brzeziński and Wisbauer, 2003)
Let R be a commutative ring with multiplicative identity and G be a group. An R-module M is a G-graded module over R if and only if M is an $R[G]$-comodule.

Clean $R[G]$-comodules

Lemma

Given R-coalgebra $R[G]$. If G is finite group with order n, then dual algebra $R[G]^* \simeq R^n$ as a ring.

Proposition

Let G be a finite group with order n and R be a field. If M is a G-graded module over R, then M is a clean $R[G]$-comodule.

Proof

As $R[G]^* \simeq R^n$ and M is an $R[G]^*$-module, then M is a module over a semi-simple ring R^n. Thus, M is a injective R^n-module. It implies M is a clean $R[G]$-module.

Proposition

Let G be a finite group with order n and R a simple ring. If M is a G-graded module over R, then M is a clean $R[G]$-comodule.

Since R^n is a semi-simple ring.

Clean $R[G]$-Comodules

Proposition

Let R be a Artinian principal ideal ring and G be a finite group. Every $R[G]$-comodule M is a clean comodule.

Since dual algebra $R[G]^* \simeq R^n$ is Artinian principal ideal ring and based on (Camillo, et.al, 2006), every module over Artinian principal ideal ring is clean.

Clean C-Comodules

Lemma

Let C be a free R-module with basis $\{x_i\}_{i=1}^n$. If C is an R-coalgebra with comultiplication

$$\Delta : C \to C \otimes_R C, x_i \mapsto x_i \otimes x_i,$$

then the dual algebra C^* is isomorphic to the ring R^n.

Clean C-Comodules

Proposition

Let R be a simple ring and C a free R-module with basis $\{x_i\}_{i=1}^{n}$. If C is an R-coalgebra with comultiplication $\Delta : C \to C \otimes_R C, x_i \mapsto x_i \otimes x_i$, then every C-comodule M is clean.

Proposition

Let R be an Artinian principal ideal ring and C is a finitely generated R-module with basis $\{x_i\}_{i=1}^{n}$. If C is an R-coalgebra with comultiplication $\Delta : C \to C \otimes_R C, x_i \mapsto x_i \otimes x_i$, then every C-comodule M is a clean comodule.

The Clean Coalgebra Opposite and Dual Algebra

Proposition

Let (C, Δ, ε) be an R-coalgebra. An R-coalgebra C is clean if and only if an R-coalgebra C^{op} is clean.

Proposition

Let A be a finitely generated and projective R-module. An R-algebra A is clean if and only if the R-coalgebra A^{*} is clean.

The Clean Coalgebra Coproduct and Coalgebra Corner

Proposition

Let a family $\{(C_\lambda, \Delta_\lambda, \varepsilon_\lambda)\}_\Lambda$ of R-coalgebas (where C_λ satisfies the α-condition for all λ) and $C = \bigoplus_\Lambda C_\lambda$. The direct sum R-coalgebra C is clean if and only if an R-coalgebra C_λ is clean for all $\lambda \in \Lambda$.

Proposition

Let (C, Δ, ε) be an R-coalgebra. If C is cocommutative a clean R-coalgebra and e is an idempotent in C^*, then R-coalgebra $e \rightharpoonup C \rightharpoonup e$ is clean.

An R-coalgebra $M_n(R)$

In ring theory, if R is a clean ring, then the ring $M_n(R)$ is clean (Han and Nicholson, 2001). Now, consider $M_n(R)$ as an R-coalgebra (see (Brzeziński and Wisbauer, 2003)). How is the condition that will make it to be clean s an R-coalgebra?

Theorem

(Brzeziński and Wisbauer, 2003)
Let P be a finitely generated projective R-module with dual basis $p_1, p_2, ..., p_n \in P$ and $\pi_1, \pi_2, ..., \pi_n \in P^*$. The R-module $P^* \otimes_R P$ is an R-coalgebra with the comultiplication and counit defined by

$$\Delta : P^* \otimes_R P \to (P^* \otimes_R P) \otimes_R (P^* \otimes_R P);$$
$$f \otimes p \mapsto \Sigma_i f \otimes p_i \otimes \pi_i \otimes p$$

and

$$\varepsilon : P^* \otimes_R P \to R, f \otimes p \mapsto f(p).$$

The Clean R-Coalgebra $P^* \otimes_R P$

Theorem

Let P be a finitely generated projective R-module with dual basis $p_1, p_2, ..., p_n \in P$ $\pi_1, \pi_2, ..., \pi_n \in P^*$. If P is a clean R-module, then the R-coalgebra $P^* \otimes_R P$ is clean.

Proof

Let P be a finitely generated R-module and $P^* = Hom_R(P, R)$ is a an R-module. Suppose P is a clean R-module. Hence, $P^* = Hom_R(P, R)$ is a f.g projective R-module. The R-coalgebra $P^* \otimes_R P$ satisfies the α-condition. It implies

$$ {}_{(P^* \otimes_R P)^*} End(P^* \otimes_R P) \simeq (P^* \otimes_R P)^*. $$

Proof

We need to prove that $(P^* \otimes_R P)^*$ is a clean ring. Since P is finitely generated, the dual algebra $P^* \otimes_R (P)$ is isomorphic to the ring $End_R(P)$ by

$$\begin{aligned}(P^* \otimes_R P)^* &= Hom_R(P^* \otimes_R P, R)\\ &\simeq Hom_R(P, Hom_R(P^*, R))\\ &\simeq Hom_R(P, P^{**})\\ &\simeq End_R(P).\end{aligned}$$

Therefore, if P is a clean R-module, $End_R P$ is a clean ring. It means $P^* \otimes_R P$ is a clean R-coalgebra.

Corollary

If R is a clean ring, then the R-coalgebra $M_n(R)$ is clean.

Proof

Based on theorem above when $P = R^n$, we have R-coalgebra $P^* \otimes_R P \cong M_n(R)$.

The Clean $P^* \otimes_R P$-Comodules

Theorem

Let P be a finitely generated projective R-module with basis $p_1, p_2, ..., p_n \in P$ and dual basis $\pi_1, \pi_2, ..., \pi_n \in P^*$. If R is a clean ring, then P is a right clean $P^* \otimes_R P$-comodule and P^* is a left clean $P^* \otimes_R P$-comodule.

Proof 1

Consider P as a right $P^* \otimes_R P$-comodule. We want to prove ${}_{(P^* \otimes_R P)^*}End(P)$ is a clean ring. Since $P^* \otimes_R P$ is a f.g projective R-module, implies that the category of

$$\mathbf{M}^{P^* \otimes_R P} =_{(P^* \otimes_R P)^*} \mathbf{M}$$

On the other hand, as a ring $(P^* \otimes_R P)^* \simeq End_R(P)$. Therefore,

$$\mathbf{M}^{P^* \otimes_R P} =_{(P^* \otimes_R P)^*} \mathbf{M} \simeq_{End_R(P)} \mathbf{M}.$$

Proof 1

We going to prove that the ring ${}_{(P^* \otimes_R P)^*}End(P) \in {}_{(P^* \otimes_R P)^*}$ **M** is clean. Based on their categories and using the Morita Context (see (Lam, 1999)) since P is generator, $R \simeq End({}_{End_R(P)}P)$ as a ring. Therefore, ${}_{(P^* \otimes_R P)^*}End(P) \simeq End_{End_R(P)}(P)$ and $End_{End_R(P)}(P) \simeq R$ as a ring. Hence, ${}_{(P^* \otimes_R P)^*}End(P) \simeq R$ as an R-module. However, R is a clean ring if and only if R is a clean R-module. Thus, ${}_{(P^* \otimes_R P)^*}End(P) \simeq R$ is a clean ring. Consequently, P is a clean $P^* \otimes_R P$-comodule.

Proof 2

Consider P^* as a right $P^* \otimes_R P$-comodule. We want to prove that $End_{(P^* \otimes_R P)^*}(P^*)$ is a clean ring. Analogue with Proof (1) we have

$$^{P^* \otimes_R P}\mathbf{M} = \mathbf{M}_{(P^* \otimes_R P)^*} \simeq \mathbf{M}_{End_R(P)}.$$

From their category we have $End_{(P^* \otimes_R P)^*}(P^*) \simeq End_{End_R P}(P^*)$. Furthermore, using Morita Theorem we have $R \simeq End_{End_R(P)}(P^*)$. Therefore,

$$End_{(P^* \otimes_R P)^*}(P^*) \simeq R$$

Consequently, if R is clean ring then $_{(P^* \otimes_R P)^*}End(P) \simeq R$ is a clean ring. In particular P^* is a left clean $P^* \otimes_R P$-comodule.

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