Non-unique factorizations in rings of integer-valued polynomials

(Joint work with Sophie Frisch and Roswitha Rissner)

Sarah Nakato

Graz University of Technology

Happy 60th birthday Prof. Blas Torrecillas

Outline

- Preliminaries on $\mathrm{Int}(D)$ and factorizations
- What is known in $\mathrm{Int}(\mathbb{Z})$
- New results

$\mathrm{Int}(D)$

Definition 1

Let D be a domain with quotient field K. The ring of integer-valued polynomials on D

$$\mathrm{Int}(D) = \{f \in K[x] \mid \forall\ a \in D, f(a) \in D\} \subseteq K[x]$$

Remark 1

1. For all $f \in K[x]$, $f = \frac{g}{b}$ where $g \in D[x]$ and $b \in D \setminus \{0\}$.
2. $f = \frac{g}{b}$ is in $\mathrm{Int}(D)$ if and only if $b \mid g(a)$ for all $a \in D$.

Examples

- $D[x] \subseteq \mathrm{Int}(D)$
- $\frac{x(x-1)}{2} \in \mathrm{Int}(\mathbb{Z})$, $\binom{x}{n} = \frac{x(x-1)(x-2)\cdots(x-n+1)}{n!} \in \mathrm{Int}(\mathbb{Z})$.

Int(D) cont'd

- Int($\mathbb{Z}$) is non-Noetherian
- Int(D) is in general not a unique factorization domain e.g., in Int($\mathbb{Z}$)

$$\frac{x(x-1)(x-4)}{2} = \frac{x(x-1)}{2}\,(x-4)$$

$$= x\,\frac{(x-1)(x-4)}{2}$$

Factorization terms

Definition 2

Let $r \in R$ be a nonzero non-unit.

1. r is said to be irreducible in R if whenever $r = ab$, then either a or b is a unit.

2. If $r = r_1 \cdots r_n$, the length of the factorization $r_1 \cdots r_n$ is the number of irreducible factors n.

3. Two factorizations of

$$r = r_1 \cdots r_n = s_1 \cdots s_m$$

are called essentially the same if $n = m$ and, after some possible reordering, $r_j \sim s_j$ for $1 \leq j \leq m$. Otherwise, the factorizations are called essentially different.

Factorization terms cont'd

- The set of lengths of r is

$$L(r) = \{n \in \mathbb{N} \mid r = r_1 \cdots r_n\}$$

where $r_1, \ldots, r_n$ are irreducibles. e.g., in $\mathrm{Int}(\mathbb{Z})$

$$\begin{aligned}\frac{x(x-2)(x^2+3)(x^2+4)}{4} &= \frac{x(x-2)(x^2+3)}{4}\,(x^2+4) \\ &= x(x-2)\,\frac{(x^2+3)(x^2+4)}{4}\end{aligned}$$

$L(r) = \{2, 3\}$

What is known in $\text{Int}(\mathbb{Z})$

Theorem 1 (Frisch, 2013)

Let $1 < m_1 \leq m_2 \leq \cdots \leq m_n \in \mathbb{N}$. Then there exists a polynomial $H \in \text{Int}(\mathbb{Z})$ with n essentially different factorizations of lengths $m_1, \ldots, m_n$.

Corollary 1

Every finite subset of $\mathbb{N}_{>1}$ is a set of lengths of an element of $\text{Int}(\mathbb{Z})$.

(Kainrath, 1999) Corollary 1 for certain monoids.

What is known in Int($\mathbb{Z}$)

Proposition 1 (Frisch, 2013)

For every $n \geq 1$ there exist irreducible elements $H, G_1, \ldots, G_{n+1}$ in Int($\mathbb{Z}$) such that $xH(x) = G_1(x) \cdots G_{n+1}(x)$.

(Geroldinger & Halter-Koch, 2006)

1. If $\theta : H \longrightarrow M$ is a transfer homomorphism, then;
   (i) $u \in H$ is irreducible in H if and only if $\theta(u)$ is irreducible in M.
   (ii) For $u \in H$, $L(u) = L(\theta(u))$
2. If u, v are irreducibles elements of a block monoid with u fixed, then $\max L(uv) \leq |u|$, where $|u| \in \mathbb{N}_{\geq 0}$.
3. Any monoid which allows a transfer homomorphism to a block monoid must have the property in 2.

Monoids which allow transfer homomorphisms to block monoids are called transfer Krull monoids.

New results

Motivation question: Are there other domains D such that $\mathrm{Int}(D)$ is not a transfer Krull monoid? YES

If D is a Dedekind domain such that;

1. D has infinitely many maximal ideals,
2. all these maximal ideals are of finite index.

Then $\mathrm{Int}(D)$ is not a transfer Krull monoid.

Examples of our Dedekind domains

1. $\mathbb{Z}$
2. $\mathcal{O}_K$, the ring of integers of a number field K

Theorem 2 (Frisch, Nakato, Rissner, 2019)

For every $n \geq 1$ there exist irreducible elements $H, G_1, \ldots, G_{n+1}$ in $\mathrm{Int}(D)$ such that $xH(x) = G_1(x) \cdots G_{n+1}(x)$.

New results

Let D be a Dedekind domain such that;

1. D has infinitely many maximal ideals,
2. all these maximal ideals are of finite index.

Theorem 3 (Frisch, Nakato, Rissner, 2019)

Let $1 < m_1 \leq m_2 \leq \cdots \leq m_n \in \mathbb{N}$. Then there exists a polynomial $H \in \mathrm{Int}(D)$ with n essentially different factorizations of lengths $m_1, \ldots, m_n$.

References

1. P.J. Cahen and J.L. Chabert, Integer-valued polynomials, volume 48 of Mathematical Surveys and Monographs. American Mathematical Society, Providence, RI, 1997.
2. A. Geroldinger and F. Halter-Koch, Non-unique factorizations, vol. 278 of Pure and Appl. Math., Chapman & Hall/CRC, Boca Raton, FL, 2006.
3. S. Frisch, A construction of integer-valued polynomials with prescribed sets of lengths of factorizations, Monatsh. Math. 171 (2013), 341 - 350.
4. S. Frisch, S. Nakato and R. Rissner, Sets of lengths of factorizations of integer-valued polynomials on Dedekind domains with finite residue fields, J. Algebra, vol. 528, pp. 231- 249, 2019

References

1. S. Frisch, Integer-valued polynomials on algebras: a survey. Actes du CIRM, 27-32, 2010.
2. S. Frisch, Integer-valued polynomials on algebras, J. Algebra, vol. 373, pp. 414- 425, 2013.
3. Nicholas J. Werner, Integer-valued polynomials on algebras: a survey of recent results and open questions. In Rings, polynomials, and modules, pages 353-375, Springer, Cham, 2017.

You are all invited to the Conference on Rings and Polynomials

When: $20^{th} - 25^{th}$ July, 2020

Where: Graz University of Technology, Graz, Austria

Website: http://integer-valued.org/rings2020/