Rings, modules, and Hopf algebras

A conference on the occasion of

Blas Torrecillas' 60th birthday

Almería, May 13-17, 2019

Claudia Menini

Heavily separable functors

Joint work with
Alessandro Ardizzoni

THANKS TO THE ORGANIZERS!!!

BUON COMPLEANNO

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BLAS!!!

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[NVV] C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen, *Separable functors applied to graded rings*. J. Algebra **123** (1989), no. 2, 397–413.

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REMARK

We were tempted to use the word "strongly" at first, instead of "heavily", but a notion of "strongly separable functor" already appeared in the literature in connection with graded rings in [CGN, Definition 3.1].

F. Castaño Iglesias, J. Gómez Torrecillas, C. Năstăsescu, *Separable functors in graded rings*. J. Pure Appl. Algebra **127** (1998), no. 3, 219–230.

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to be explained at the end of the talk!

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We now recall the well-known:

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"Created during the algebra seminar of F. Van Oystaeyen at Cortona (Italy), Summer 1988 and it is based upon contributions from the following members of M. D. Rafael :

- M. Saorin (Univ. de Murcia, Spain)
- D. Herbera (Univ. Autonoma de Barcelona, Spain)
- R. Colpi (Univ. di Padova, Italy)
- A. Del Rio Mateos (Univ. de Murcia, Spain)
- F. Van Oystaeyen (UIA, University of Antwerp, Belgium)
- A. Giaquinta (Univ. of Pennsylvania, USA)
- E. Gregorio (Univ. di Padova, Italy)
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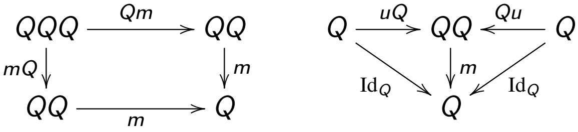

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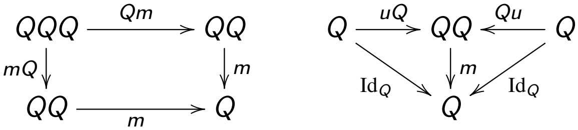

An algebra over a monad $\mathbb{Q} = (Q, m, u)$ (or simply a $\mathbb{Q}$-*algebra*) is a pair (X, μ) where $X \in \mathscr{C}$ and $\mu : QX \to X$ is a morphism in $\mathscr{C}$ s.t.

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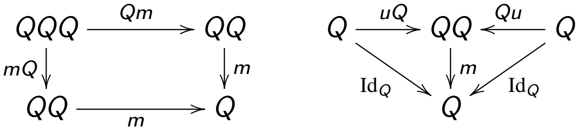

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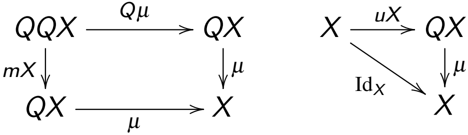

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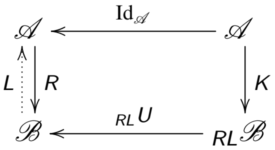

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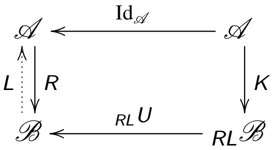

where

- ${}_{RL}U$ is the forgetful functor: ${}_{RL}U(A, \mu) := A$ and ${}_{RL}Uf := f$.
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Here $\mathscr{B}^{LR}$ denotes the Eilenberg-Moore category of the comonad $(LR, L\eta R, \varepsilon)$.

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[LMW] M. Livernet, B. Mesablishvili, R. Wisbauer, *Generalised bialgebras and entwined monads and comonads.* J. Pure Appl. Algebra **219** (2015), no. 8, 3263–3278.

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[Br] T. Brzeziński, *The structure of corings: induction functors, Maschke-type theorem, and Frobenius and Galois-type properties.* Algebr. Represent. Theory **5** (2002), no. 4, 389–410.

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[NVV] C. Năstăsescu, M. Van den Bergh, F. Van Oystaeyen, *Separable functors applied to graded rings*. J. Algebra **123** (1989), no. 2, 397–413

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Note that for an element

$$e := \sum_i a_i \otimes_R b_i \in S \otimes_R S$$

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PROPOSITION

S/R is h-separable $\Leftrightarrow$ S/R has a h-separability idempotent.

Let $\varphi : R \to S$ be a ring homomorphism.
Then $\mathscr{C} := S \otimes_R S$ is an S-coring, called the Sweedler coring, where the coproduct is

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e is a h-separability idempotent $\Leftrightarrow$ e is a group-like element in the Sweedler's coring $\mathscr{C} := S \otimes_R S$ such that $se = es$ for every $s \in S$

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If these equivalent conditions hold true then S/R is h-separable.

Proof

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[St] B. Stenström, *Rings of quotients* Die Grundlehren der Mathematischen Wissenschaften, Band **217**. An introduction to methods of ring theory. Springer-Verlag, New York-Heidelberg, 1975.

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[DI] F. DeMeyer, E. Ingraham, *Separable algebras over commutative rings*. Lecture Notes in Mathematics, Vol. **181** Springer-Verlag, Berlin-New York 1971

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We have so proved that $S \subseteq Z(S)$ and hence S is commutative.

Now, we compute

$$\sum_i a_i \otimes_R b_i = \sum_{i,j} a_i a_j b_j \otimes_R b_i \overset{S=Z(S)}{=} \sum_{i,j} a_j a_i b_j \otimes_R b_i = \sum_{i,j} a_i b_j \otimes_R b_i a_j \overset{(2)}{=} 1_S \otimes_R 1_S$$

We conclude by previous Proposition.
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See [AM1, Remark 1.2].

[AM1]A. Ardizzoni and C. Menini, *Adjunctions and Braided Objects*, J. Algebra Appl. **13**(06) (2014), 1450019 (47 pages).

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But this is not the case. **This happens only if all objects are isomorphic to the unit object 1**,

[AM2]A. Ardizzoni and C. Menini, *Milnor-Moore Categories and Monadic Decomposition*, J. Algebra **448** (2016), 488-563.

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As a particular case, we get that the functor $T : \mathrm{Vec}_{\Bbbk} \to \mathrm{Alg}_{\Bbbk}$ is separable but not heavily separable.

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For any $\mathbb{B} := (B, m_B, u_B, \Delta_B, \varepsilon_B) \in \mathrm{Bialg}(\mathscr{M})$, $P(\mathbb{B})$ is defined via the equalizer

$$P(\mathbb{B}) \xrightarrow{\xi\mathbb{B}} B \underset{(B \otimes u_B) r_B^{-1} + (u_B \otimes B) l_B^{-1}}{\overset{\Delta_B}{\rightrightarrows}} B \otimes B$$

Let

$\widetilde{\eta}$ and $\widetilde{\varepsilon}$ denote the unit and the counit of this adjunction.

[AM1] A. Ardizzoni and C. Menini, *Adjunctions and Braided Objects*, J. Algebra Appl. **13**(06) (2014), 1450019 (47 pages).

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Then

$$\gamma \circ \widetilde{\eta} = \mathrm{Id} \text{ and } \gamma\gamma = \gamma \circ P\widetilde{\varepsilon}\widetilde{T}$$

i.e. $\widetilde{T}$ **is heavily separable via** γ.