Rings, modules, and Hopf algebras

A conference on the occasion of

Blas Torrecillas’ 60th birthday

Almería, May 13-17, 2019

Claudia Menini

Heavily separable functors

Joint work with
Alessandro Ardizzoni
THANKS TO THE ORGANIZERS!!!
BUON COMPLEANNO
BUON COMPLEANNO
BLAS!!!
DEFINITIONS

For every function $F : B \to A$ we set $F(X,Y) : \text{Hom}_B(X,Y) \to \text{Hom}_A(FX,FY) : f \mapsto Ff$.

Recall that $F$ is called separable (see [NVV]) if it is a split natural monomorphism i.e. there is a natural transformation $P : B \to A$ such that $P_X,Y \circ F_X,Y = \text{Id}$ for every $X,Y \in B$. 

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\begin{array}{c}
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where the vertical arrows are the obvious compositions. On elements the above diagram means that $P_X, Z(f \circ g) = P_Y, Z(f) \circ P_X, Y(g)$.

**REMARK** We were tempted to use the word “strongly” at first, instead of “heavily”, but a notion of “strongly separable functor” already appeared in the literature in connection with graded rings (Corros, 1998, no. 3, 2192–230).


C. Menini (University of Ferrara) May 10, 2019 5 / 43
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Why h-separable functors?
Why \( h \)-separable functors?

to be explained at the end of the talk!
A full and faithful functor is h-separable.

In fact, if \( F : B \to A \) is full and faithful, we have that the canonical map

\[
F : \text{Hom}_B(X, Y) \to \text{Hom}_A(FX, FY)
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is invertible so that we can take

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P_{X, Y} = F^{-1}X, Y : \text{Hom}_A(FX, FY) \to \text{Hom}_B(X, Y).
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Since \( F \) is a functor, the following diagram commutes

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1) \(L\) is separable if and only if \(\eta\) is a split mono, i.e. if there is a natural transformation \(\gamma : RL \to \text{Id}_\mathcal{B}\) such that \(\gamma \circ \eta = \text{Id}\).
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M. D. Rafael, Separable Functions Revisited, Comm. Algebra 18 (1990), 1445–1459.

"Created during the algebra seminar of F. Van Oystaeyen at Cortona (Italy), Summer 1988 and it is based upon contributions from the following members of M. D. Rafael: M. Sanin (Univ. de Murcia, Spain) D. Herrero (Univ. Autonoma de Barcelona, Spain) R. Colpi (Univ. di Padova, Italy) A. Del Rio Mateos (Univ. de Murcia, Spain) F. Van Oystaeyen (UIA, University of Antwerp, Belgium) A. Giaquinta (Univ. of Pennsylvania, USA) E. Gregorio (Univ. di Padova, Italy)."

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\begin{align*}
\xymatrix{
QQQ & \ar[r]^{Qm} & QQ \\
QQQ & \ar[r]^{Qm} & QQ \\
QQ & \ar[r]^m & Q \\
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}
\end{align*}
$$
Recall that a monad on a category $\mathcal{C}$ is a triple $\mathcal{Q} := (Q, m, u)$, where

- $Q : \mathcal{C} \to \mathcal{C}$ is a functor,
- $m : QQ \to Q$ and $u : \text{Id}_\mathcal{C} \to Q$ are functorial morphisms s.t.

$$
\begin{align*}
QQQ \xrightarrow{Qm} QQ & \quad Q \xrightarrow{uQ} QQ \\
\downarrow mQ & \quad \downarrow m \\
QQ \xrightarrow{m} Q & \quad Q \\
\end{align*}
$$

An algebra over a monad $\mathcal{Q} = (Q, m, u)$ (or simply a $\mathcal{Q}$-algebra) is a pair $(X, \mu)$ where $X \in \mathcal{C}$ and $\mu : QX \to X$ is a morphism in $\mathcal{C}$ s.t.
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QQQ & \xrightarrow{Qm} & QQ \\
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QQ & \xrightarrow{m} & Q \\
\end{array}
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\[
\begin{array}{ccc}
QQX & \xrightarrow{Q\mu} & QX \\
\downarrow mX & & \downarrow \mu \\
QX & \xrightarrow{\mu} & X \\
\end{array}
\]

\[
\begin{array}{ccc}
X & \xrightarrow{uX} & QX \\
\downarrow \text{Id}_X & & \downarrow \mu \\
X & & X \\
\end{array}
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$\mathcal{Q}$-algebras and their morphisms form the so-called **Eilenberg-Moore category** $\mathcal{Q}^{\mathcal{C}}$ of the monad $\mathcal{Q}$. 
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Q & \xleftarrow{\text{Id}_Q} & Q
\end{array}
\quad
\begin{array}{ccc}
Q & \xleftarrow{Qu} & Q \\
\downarrow m & & \downarrow \text{Id}_Q \\
Q & \xrightarrow{\text{Id}_Q} & Q
\end{array}
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Denote by \(RL\mathcal{B}\) the category of algebras over this monad.

We have a commutative diagram

\[
\begin{array}{ccc}
\mathcal{A} & \xleftarrow{\text{Id}_\mathcal{A}} & \mathcal{A} \\
\downarrow{L} & & \downarrow{R} \\
\mathcal{B} & \xleftarrow{RLU} & RL\mathcal{B} \\
\end{array}
\]

where

- \(RLU\) is the **forgetful** functor: \(RLU(A, \mu) := A\) and \(RLUf := f\).
- \(K\) is **comparison** functor: \(KA := (RA, R\varepsilon A)\) and \(Kf := Rf\).
PROPOSITION

Let \((L, R)\) be an adjunction.

1) \(L\) is h-separable \(\iff\) \(U : RL \to B\) is a split natural epimorphism, i.e. there is \(\Gamma : B \to RL\) such that \(U \circ \Gamma = \text{Id}_B\).

2) \(R\) is h-separable \(\iff\) \(U : B \to LR\) is a split natural epimorphism, i.e. there is \(\Gamma : B \to LR\) such that \(U \circ \Gamma = \text{Id}_B\).

Here \(B_{LR}\) denotes the Eilenberg-Moore category of the comonad \((LR, \eta_R, \varepsilon)\).
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Here \(\mathcal{B}^{LR}\) denotes the Eilenberg-Moore category of the comonad \((LR, L\eta R, \varepsilon)\).
Proof

We just prove 1. By h-version of RAFAEL THEOREM, $L$ is h-separable if and only if there is a natural transformation $\gamma: RL \to Id_B$ such that $\gamma \circ \eta = Id$ and $\gamma \circ R \varepsilon_L = \gamma \circ R \varepsilon_L$. This means that, for every $B \in B$, we have $\Gamma_B := (B, \gamma_B) \in RL_B$.

Moreover any morphism $f: B \to C$ fulfills $f \circ \gamma_B = \gamma_C \circ RLf$ by naturality of $\gamma$. This means that $f$ induces a morphism $\Gamma_f: \Gamma_B \to \Gamma_C$ such that $U \circ \Gamma_f = f$.

We have defined a functor $\Gamma: B \to B_{RL}$ such that $U \circ \Gamma = Id_B$. 

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Moreover any morphism $f: B \to C$ fulfills

$$f \circ \gamma B = \gamma C \circ RLf$$

by naturality of $\gamma$. This means that $f$ induces a morphism

$$\Gamma f : \Gamma B \to \Gamma C$$

such that

$$U\Gamma f = f.$$ 

We have so defined a functor

$$\Gamma : \mathcal{B} \to \mathcal{B}_{RL}$$

such that $U \circ \Gamma = \text{Id}_\mathcal{B}$. 

Conversely, let $\Gamma$ be a functor such that $\Gamma : B \to B_{RL}$ such that $U \circ \Gamma = \text{Id}_B$. Then, for every $B \in B$, we have that $\Gamma_B = (B, \gamma_B)$ for some morphism $\gamma_B : RL_B \to B$. Since $\Gamma_B \in B_{RL}$ we must have that $\gamma_B \circ \eta_B = B$ and $\gamma_B \circ RL \gamma_B = \gamma_B \circ R \varepsilon_L B$. Given a morphism $f : B \to C$, we have that $\Gamma_f : \Gamma_B \to \Gamma_C$ is a morphism in $RL_B$, which means that $f \circ \gamma_B = \gamma_C \circ RL f$. i.e. $\gamma_B := (\gamma_B)_B \in B_{RL}$ is a natural transformation. By the foregoing $\gamma \circ \eta = \text{Id}$ and $\gamma \circ RL \gamma = \gamma \circ R \varepsilon_L$. 
Conversely, let $\Gamma$ be a functor such that

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\Gamma : \mathcal{B} \to \mathcal{B}_{\text{RL}} \text{ such that } U \circ \Gamma = \text{Id}_{\mathcal{B}}.
\]

Then, for every \( B \in \mathcal{B} \), we have that

\[
\Gamma_B = (B, \gamma_B) \text{ for some morphism } \gamma_B : \mathcal{B}_{\text{RL}} \to B.
\]

Since \( \Gamma_B \in \mathcal{B}_{\text{RL}} \) we must have that

\[
\gamma_B \circ \eta_B = B \text{ and } \gamma_B \circ \text{RL} \gamma_B = \gamma_B \circ \text{R} \varepsilon_{LB}.
\]

Given a morphism \( f : B \to C \), we have that

\[
\Gamma_f : \Gamma_B \to \Gamma_C \text{ is a morphism in } \mathcal{B}_{\text{RL}},
\]

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\[
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Conversely, let $\Gamma$ be a functor such that

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Then, for every $B \in \mathcal{B}$, we have that

$$\Gamma B = (B, \gamma_B) \text{ for some morphism } \gamma_B : RLB \to B.$$
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Given a morphism $f : B \to C$, we have that $\Gamma f : \Gamma B \to \Gamma C$ is a morphism in $RL\mathcal{B}$.
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Then, for every $B \in \mathcal{B}$, we have that

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Given a morphism $f : B \to C$, we have that $\Gamma f : \Gamma B \to \Gamma C$ is a morphism in $\mathcal{RL} \mathcal{B}$, which means that

$$f \circ \gamma B = \gamma C \circ RL f$$

i.e.

$$\gamma := (\gamma B)_{B \in \mathcal{B}} \text{ is a natural transformation.}$$
Conversely, let $\Gamma$ be a functor such that

$$\Gamma : \mathcal{B} \to \mathcal{B}_{RL} \text{ such that } U \circ \Gamma = \text{Id}_B.$$ 

Then, for every $B \in \mathcal{B}$, we have that

$$\Gamma B = (B, \gamma B) \text{ for some morphism } \gamma B : RLB \to B.$$ 

Since $\Gamma B \in \mathcal{B}_{RL}$ we must have that

$$\gamma B \circ \eta B = B \text{ and } \gamma B \circ RL\gamma B = \gamma B \circ R\varepsilon LB.$$ 

Given a morphism $f : B \to C$, we have that $\Gamma f : \Gamma B \to \Gamma C$ is a morphism in $RL\mathcal{B}$, which means that

$$f \circ \gamma B = \gamma C \circ RLf$$

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By the foregoing

$$\gamma \circ \eta = \text{Id} \text{ and } \gamma \circ RL\gamma = \gamma \circ R\epsilon L.$$
COROLLARY

1) Assume that $R$ is strictly monadic (i.e. the comparison functor $K: A \to RLB$ is an isomorphism of categories). Then $L$ is h-separable $\iff R$ is a split natural epimorphism.

2) Assume that $L$ is strictly comonadic (i.e. the co-comparison functor $K^\text{co}: B \to A LR$ is an isomorphism of categories). Then $R$ is h-separable $\iff L$ is a split natural epimorphism.

REMARK Later we will use 1) of this Corollary to obtain that the tensor algebra functor $T: M \to \text{Alg}(M)$ is separable but not h-separable.

C. Menini (University of Ferrara)
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Later we will use 1) of this Corollary to obtain that the tensor algebra functor $T : M \to \text{Alg}(M)$ is separable but not h-separable.
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REMARK
Later we will use 1) of this Corollary to obtain that the tensor algebra functor

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We just prove 1), the proof of 2) being similar.

Since the comparison functor $K: A \to B$ is an isomorphism of categories and $U \circ K = R$ we get that $R$ is a split natural epimorphism $\Leftrightarrow U$ is a split natural epimorphism.

By previous Proposition, $U$ is a split natural epimorphism $\Leftrightarrow L$ is h-separable.
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Following [LMW, Section 4] we say that an augmentation for a monad 
\((M, m : MM \to M, \eta : \text{Id} \to M)\) is a natural transformation 
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Following [LMW, Section 4] we say that an **augmentation** for a monad 
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Dually a **grouplike morphism** for a comonad \((C, \Delta : C \to CC, \varepsilon : C \to \text{Id})\) is a natural transformation

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Let \((L, R, \eta, \varepsilon)\) be an adjunction with \(L : \mathcal{B} \to \mathcal{A}\).

a) \(L\) is h-separable \(\iff\) the monad \((RL, R\varepsilon L, \eta)\) has an augmentation.
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**h-version of RAFAEL THEOREM**

Let \((L, R, \eta, \varepsilon)\) be an adjunction with \(L : \mathcal{B} \to \mathcal{A}\).

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b) \(R\) is h-separable \(\iff\) the comonad \((LR, L\eta R, \varepsilon)\) has a grouplike morphism.
Consider an $S$-coring $C$
Consider an $S$-coring $C$ and its set of invariant elements.

Let $C$ be the category of left $C$-comodules.

In [Br, Theorem 3.3], Brzeziński proved that the induction functor $R := C \otimes S(-) : S\text{-Mod} \to C\text{M}$ is separable $\iff$ there is an invariant element $e \in CS$ such that $\varepsilon C(e) = 1$.

We prove that the induction functor $R := C \otimes S(-) : S\text{-Mod} \to C\text{M}$ is h-separable $\iff$ $C$ has an invariant group-like element.

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REMARK

Let $C$ be an $S$-co-ring.
We recall that, by [Br, Lemma 5.1], if $S$ is a left $C$-comodule via $\rho_S: S \to C \otimes SS$
then $g = \rho_S(1_S)$ is a group-like element of $C$.
Conversely if $g$ is a group-like element of $C$, then $S$ is a left $C$-comodule via $\rho_S: S \to C \otimes SS$
$s \mapsto (s \cdot g) \otimes 1_S$.
Moreover, if $g$ is a group-like element of $C$, then by [Br, page 404], $g$ is an invariant element of $C$ $\iff S = S^{coC} = \{ s \in S \mid sg = gs \}$.
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PROPOSITION

Let \( \varphi : R \to S \) be a ring homomorphism. Then the induction function \( \varphi^* : S \otimes_R (-) : R\text{-mod} \to S\text{-mod} \) is h-separable if and only if there is a ring homomorphism \( E : S \to R \) such that \( E \circ \varphi = \text{Id} \).
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Recall that $S/R$ is said to be separable if the multiplication map

$$
\mu : S \otimes_R S \to S \quad s \otimes s' \mapsto ss'
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is a split $S$-bimodule surjective homomorphism. Let $\varphi^* : S\text{-Mod} \to R\text{-Mod}$ be the restriction of scalar function. Then it is well-known that $\varphi^* : S\text{-Mod} \to R\text{-Mod}$ is separable (see [NVV, Proposition 1.3]) if and only if $S/R$ is separable and $S/R$ has a separability idempotent, where an element $\sum_i a_i \otimes_R b_i \in S \otimes_R S$ is a separability idempotent if $\sum_i a_i b_i = 1$. 
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$\sum_i s a_i \otimes_R b_i = \sum_i a_i \otimes_R b_i$ for every $s \in S$. 

[NVV] C. Nastasescu, M. Van den Bergh, F. Van Oystaeyen, 
\textit{Separable functors applied to graded rings}.

C. Menini (University of Ferrara) May 10, 2019 22 / 43
Let $\varphi : R \to S$ be a ring homomorphism. Recall that $S/R$ is said to be 
\textbf{separable} if the multiplication map

$$\mu : S \otimes_R S \to S$$
$$s \otimes_R s' \mapsto ss'$$

is a split $S$-bimodule surjective homomorphism. Let
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$\phi^* : S\text{-Mod} \to R\text{-Mod}$ is separable \textit{(see [NVV, Proposition 1.3])} $\iff S/R$ is separable.
Let $\varphi : R \to S$ be a ring homomorphism. Recalling that $S/R$ is said to be separable if the multiplication map

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is separable (see [NVV, Proposition 1.3]) if and only if $S/R$ is separable.
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Let $\phi: \mathbb{R} \to \mathbb{S}$ be a ring homomorphism.

1) $\mathbb{S}/\mathbb{R}$ is $h$-separable if the functor $\phi^*: \mathbb{S}\text{-Mod} \to \mathbb{R}\text{-Mod}$ is $h$-separable.

2) A heavy separability idempotent (short) of $\mathbb{S}/\mathbb{R}$ is an element $\sum_i a_i \otimes_R b_i \in \mathbb{S} \otimes_R \mathbb{S}$ such that $\sum_i a_i \otimes_R b_i$ is a separability idempotent, i.e.

$$\sum_i a_i b_i = 1, \quad \sum_i s a_i \otimes_R b_i = \sum_i a_i \otimes_R b_i$$

for every $s \in \mathbb{S}$, which moreover fulfills

$$\sum_i j a_i \otimes_R b_i a_j \otimes_R b_j = \sum_i a_i \otimes_R 1_S \otimes_R b_i.$$

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We prove

Let $\varphi : R \to S$ be a ring homomorphism. Then $C := S \otimes_R S$ is an $S$-coring, called the Sweedler coring, where the coproduct is

$$\Delta_C : S \otimes_R S \to S \otimes_R S \otimes_R S \otimes_R S \ni x \otimes y \mapsto x \otimes R_1 S \otimes S_1 R \otimes y$$

and the counit is $\epsilon_C : S \otimes_R S \to S \otimes_R S \ni x \otimes y \mapsto xy$.

Note that for an element $e := \sum a_i \otimes R b_i \in S \otimes_R S$ we have $e$ is a $h$-separability idempotent $\iff e$ is a group-like element in the Sweedler's coring $C := S \otimes_R S$ such that $se = es$ for every $s \in S$ i.e. which is an invariant.

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the induction functor $R := C \otimes_S (\_): S\text{-Mod} \to C\mathcal{M}$ is h-separable $\iff$

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Proposition

Let $\phi: R \to S$ be a ring homomorphism. The following are equivalent.

1. The map $\phi$ is a ring epimorphism (i.e. an epimorphism in the category of rings);
2. The multiplication $m: S \otimes_R R \to S$ is invertible;
3. $1_S \otimes_R 1_S$ is a separability idempotent for $S/R$;
4. $s \otimes_R 1_S = 1_S \otimes_R s$ for every $s \in S$;
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If these equivalent conditions hold true then $S/R$ is h-separable.
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Then

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and

$$(h \circ m)(s' \otimes_R s) = s's \otimes_R 1_S =$$

$$= (m \otimes_R S)(s' \otimes_R s \otimes_R 1_S) =$$

$$= (m \otimes_R S)(s' \otimes_R 1_S \otimes_R s) = s' \otimes_R s$$

and hence $h \circ m = \text{Id}$. Hence $m$ is invertible.

By a previous Proposition, (5) implies that $S/R$ is $h$-separable.
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and the injectivity of $m$ we deduce (4).

(4) $\Rightarrow$ (2) Let

$$h: S \rightarrow S \otimes_R S$$

$$s \mapsto s \otimes_R 1_S.$$

Then

$$m \circ h = \text{Id}$$

and

$$(h \circ m)(s' \otimes_R s) = s's \otimes_R 1_S =$$

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By a previous Proposition, (5) implies that $S/R$ is h-separable.
EXAMPLES

1) Let $S$ be a multiplicative closed subset of a commutative ring $R$. Then the canonical map $\phi: R \to S^{-1}R$ is a ring epimorphism. More generally we can consider a perfect right localization of $R$ as in [St, page 229].

2) Consider the ring of matrices $M_n(R)$ and the ring $T_n(R)$ of $n \times n$ upper triangular matrices over a ring $R$. Then the inclusion $\phi: T_n(R) \to M_n(R)$ is a ring epimorphism (Exercise).

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Then the following are equivalent.
1. \( S/\!\!/R \) is h-separable.
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Proof

Let \( \sum_i a_i \otimes R b_i \) be an h-separability idempotent. Since \( \phi(R) \subseteq Z(S) \), the map \( \tau: A \otimes R A \to A \otimes R A \), \( \tau(a \otimes R b) = b \otimes R a \), is well-defined and left \( R \)-linear. Hence we can apply \( (m \otimes R S) \circ (A \otimes R \tau) \) on both sides of \( \sum_j \)

\[ \sum_j a_j \otimes R b_j = \sum_j a_j \otimes R 1 \cdot S \otimes R b_j (1) \]

together with the equality \( \sum_i a_i b_i = 1 \) to get

\[ \sum_t, j a_t b_j \otimes R b_t a_j = 1 \cdot S \otimes R 1 \cdot S \]

(2)

By \( \sum_i \) sa_i \otimes R b_i = \( \sum_i a_i \otimes R b_i \) and using \( \tau \) we get that \( \sum_t a_t sb b_t \in Z(S) \), for all \( s \in S \).

Using this fact we have

\[ s = 1 \cdot S \cdot 1 \cdot S \cdot s \]

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We have so proved that \( S \subseteq Z(S) \) and hence \( S \) is commutative.
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$$
s = 1_S \cdot 1_S \cdot s \overset{(2)}{=} \sum_{i,j} a_i b_j b_i a_j s = \sum_{i,j} a_i (b_j) b_i (a_j) s (1_S) \overset{(1)}{=} \sum_{i,j,t} a_i b_j b_i a_t s b_t a_j
$$

$$
= \sum_{i,j,t} a_i b_j b_i (a_t s b_t) a_j = \sum_{i,j,t} a_i b_j b_i a_j (a_t s b_t) \overset{(2)}{=} \sum_t a_t s b_t \in Z(S).
$$
Proof

(1) ⇒ (2) Let $\sum_i a_i \otimes_R b_i$ be an h-separability idempotent. Since
$\varphi(R) \subseteq Z(S)$, the map $\tau : A \otimes_R A \rightarrow A \otimes_R A, \tau(a \otimes_R b) = b \otimes_R a$, is well-defined and left $R$-linear. Hence we can apply $(m \otimes_R S) \circ (A \otimes_R \tau)$ on both sides of

$$
\sum_{j,t} a_t \otimes_R b_t a_j \otimes_R b_j = \sum_j a_j \otimes_R 1_S \otimes_R b_j \tag{1}
$$

together with the equality $\sum_i a_i b_i = 1$ to get

$$
\sum_{t,j} a_t b_j \otimes_R b_t a_j = 1_S \otimes_R 1_S. \tag{2}
$$

By $\sum_i sa_i \otimes_R b_i = \sum_i a_i \otimes_R b_i s$ and using $\tau$ we get that $\sum_t a_t s b_t \in Z(S)$, for all $s \in S$. Using this fact we have

$$
s = 1_S \cdot 1_S \cdot s = \sum_{i,j} a_i b_j b_i a_j s = \sum_{i,j} a_i (b_j) b_i (a_j) s (1_S) = \sum_{i,j,t} a_i b_j b_i a_t s b_t a_j \tag{1}
$$

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We have so proved that $S \subseteq Z(S)$ and hence $S$ is commutative.
Now, we compute

\[ \sum_i a_i \otimes_R b_i = \sum_{i,j} a_i a_j b_j \otimes_R b_i \stackrel{S=Z(S)}{=} \sum_{i,j} a_j a_i b_j \otimes_R b_i = \sum_{i,j} a_i b_j \otimes_R b_i a_j \stackrel{(2)}{=} 1_s \otimes_R 1_s. \]

We conclude by previous Proposition.

(2) \implies (1) It follows by previous Proposition.
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**Proposition**

Let $A$ be a $h$-separable algebra over a field $k$. 

By previous theorem, the unit $u: k \rightarrow A$ is a ring epimorphism. By previous proposition, we have that $A \otimes_k A \sim A$ via multiplication. Since $A$ is $h$-separable over $k$ it is in particular separable over $k$ and hence it is finite-dimensional. Thus, from $A \otimes_k A \sim A$ we deduce that $A$ has either dimensional one or zero over $k$. 
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\( C / R \) is separable but not h-separable. In fact, by Proposition above, \( C / R \) is not h-separable. On the other hand \( e = \frac{1}{2} (1 \otimes R_1 - i \otimes R_i) \) is a separability idempotent (it is the only possible one). It is clear that \( e \) is not a h-separability idempotent.
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Let us consider the adjunction

\[(T, \Omega)\]

where \(T : M \rightarrow \text{Alg}(M)\) is the tensor algebra functor and \(\Omega : \text{Alg}(M) \rightarrow M\) is the forgetful functor.

Let \(V \in M\). By construction, \(\Omega TV = \bigoplus_{n \in \mathbb{N}} V \otimes n\).

Denote by \(\alpha_n V : V \otimes n \rightarrow \Omega TV\) the canonical inclusion.

The unit of the adjunction \((T, \Omega)\) is \(\eta : \text{Id}_M \rightarrow \Omega T\) defined by \(\eta_V := \alpha_1 V\).
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while the counit $\varepsilon : T\Omega \to \text{Id}$ is uniquely defined by the equality

$$\varepsilon = m^n - 1$$

for every $n \in \mathbb{N}$ where $m^n - 1 : A \otimes n \to A$ denotes the iterated multiplication of an algebra $(A, m, u)$ defined by

$$m^0 = \text{Id}_A$$

and for $n \geq 2$, $m^n - 1 = m \circ (m^{n-2} \otimes A)$. 

See [AM1, Remark 1.2].

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Since the functor $\Omega$ is strictly monadic, by the foregoing, we have that

$T$ is heavily separable if and only if

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But this is not the case. This happens only if all objects are isomorphic to the unit object $1$,

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\[ \omega \circ \eta = \text{Id}, \]
so that the functor

$$T : M \to \text{Alg}(M)$$

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Conclusion: the tensor functor

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As a particular case, we get that the functor

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\[\widetilde{T} : \mathcal{M} \rightarrow \text{Bialg}(\mathcal{M})\]

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\[ P : \text{Bialg}(\mathcal{M}) \to \mathcal{M} \] is the "primitive elements functor".

For any \( B := (B, m_B, u_B, \Delta_B, \varepsilon_B) \in \text{Bialg}(\mathcal{M}) \), \( P(B) \) is defined via the equalizer

\[ P(B) \xrightarrow{\xi_B} B \xrightarrow{\Delta_B} B \otimes B \]

\[ (B \otimes u_B) r_B^{-1} + (u_B \otimes B) l_B^{-1} \]

Let

\( \tilde{\eta} \) and \( \tilde{\varepsilon} \) denote the unit and the counit of this adjunction.

---

Set

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Then

\[ \gamma \circ \tilde{\eta} = \text{Id} \quad \text{and} \quad \gamma \gamma = \gamma \circ P\tilde{\varepsilon} \tilde{T} \]

i.e. \( \tilde{T} \) is heavily separable via \( \gamma \).