Lattice theory and module theory

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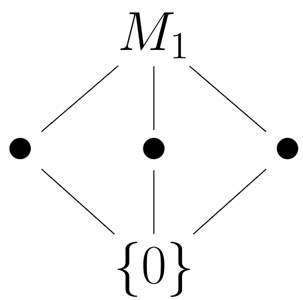
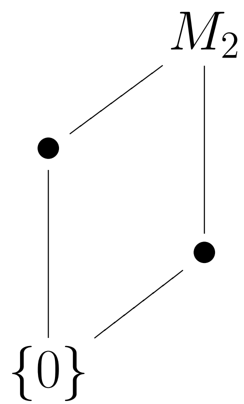
Almería, 13/05/2019. Rings, modules, and Hopf algebras A conference on the occasion of Blas Torrecillas' 60th birthday Almería, May 13-17, 2019

1 Lattice decomposition

Let M be a left R–module and $\mathcal{L}(M)$ the lattice of all submodules of M.
If we consider the abelian group $M_1 = \mathbb{Z}_2 \times \mathbb{Z}_2$, the lattice of subgroups is:



On the other hand, the lattice of subgroups of $M_2 = \mathbb{Z}_2 \times \mathbb{Z}_3 \cong \mathbb{Z}_6$ is:



Lattices.

If $\mathcal{L}_1$ and $\mathcal{L}_2$ are lattices, the product, $\mathcal{L}_1 \times \mathcal{L}_2$ is defined as

(1) $(a_1, b_1) \leq (a_2, b_2)$ if $a_1 \leq a_2$ and $b_1 \leq b_2$, or

(2) $(a_1, b_2) \wedge (a_2, b_2) = (a_1 \wedge a_2, b_1 \wedge b_2)$, and $(a_1, b_2) \vee (a_2, b_2) = (a_1 \vee a_2, b_1 \vee b_2)$

Therefore, $\mathcal{L}(\mathbb{Z}_2 \oplus \mathbb{Z}_3) \cong \mathcal{L}(\mathbb{Z}_2) \times \mathcal{L}(\mathbb{Z}_3)$ and $\mathcal{L}(\mathbb{Z}_2 \oplus \mathbb{Z}_2) \ncong \mathcal{L}(\mathbb{Z}_2) \times \mathcal{L}(\mathbb{Z}_2)$

Problem.

Is it possible to determine when the lattice of a left R–module is the direct product of the lattices of two non–zero submodules?

Product of lattices

If $\mathcal{L}$ is a bounded lattice which is the product of two bounded lattices: $\mathcal{L} = \mathcal{L}_1 \times \mathcal{L}_2$, then

$$0 = (0, 0) \quad \text{and} \quad 1 = (1, 1).$$

One can identify $\mathcal{L}_1$ with $\{(a, x) \mid \ x \in \mathcal{L}_2 \text{ (fixed)}, a \in \mathcal{L}_1\}$. In particular, if we take $x = 0$, or $x = 1$, we may have better identifications.
The following are lattice maps, and they don't apply the top in the top.

$$j_1 : \mathcal{L}_1 \longrightarrow \mathcal{L},\ j_1(a) = (a, 0),$$
$$j_2 : \mathcal{L}_2 \longrightarrow \mathcal{L},\ j_2(b) = (0, b).$$

But each element (a, b) of $\mathcal{L}$ can be written as $(a, b) = (a, 0) \vee (0, b)$.

The image of j_1 is the interval $[0,(1,0)]$, we call $e_1 = (1,0)$.
The image of j_2 is the interval $[0,(0,1)]$, we call $e_2 = (0,1)$.
These elements e_1 and e_2 are special as they satisfy:

(1) $e_1 \vee e_2 = 1$ and $e_1 \wedge e_2 = 0$. They are complemented.

(2)
$$e_1 \wedge (a,b) = (1,0) \wedge (a,b) = (a,0), \text{ and}$$
$$e_1 \vee (a,b) = (1,0) \vee (a,b) = (1,b).$$

(3)
$$e_1 \vee [(a_1,b_1) \wedge (a_2,b_2)] = e_1 \vee (a_1 \wedge b_1, a_2 \wedge b_2) = (1, b_1 \wedge b_2)$$
$$[e_1 \vee (a_1,b_1)] \wedge [e_1 \vee (a_2,b_2)] = (1,b_1) \wedge (1,b_2)$$

This means e_1 distributes and the same for e_2. They are distributive elements in the lattice $\mathcal{L}$.

Result.

There exists a bijective correspondence between:

(a) Decompositions of $\mathcal{L}$ as a product of bounded lattices.
(b) Elements $e \in \mathcal{L}$ which are distributive and complemented.

Case of modules

Let M be a left R–module, and $\mathcal{L}(M)$ the lattice of submodules, to get a decomposition of $\mathcal{L}(M)$ we need a direct summand $N \subseteq M$ (= a complemented element in $\mathcal{L}(M)$) and in addition, we need that N is distributive in $\mathcal{L}(M)$, or equivalently,

$$\begin{array}{l} N + (X \cap Y) = (N + X) \cap (N + Y) \text{ and} \\ N \cap (X + Y) = (N \cap X) + (N \cap Y), \text{ for any } X, Y \subseteq M. \end{array}$$

Result. Distributive submodules can be characterized using subfactors. A subfactor of a left R–module X is a submodule of a homomorphic image of X.

For any submodule $N \subseteq M$ the following are equivalent:

(a) For every $H \subseteq M$ we have that $N/(N \cap H)$ and $H/(N \cap H)$ have no non–zero isomorphic subfactors.

(b) For every $H \subseteq M$ we have that $N/(N \cap H)$ and $H/(N \cap H)$ have no simple isomorphic subfactors.

(c) $N \subseteq M$ is distributive in $\mathcal{L}(M)$.

If, in addition, we impose to N the condition to be complemented, then the following are equivalent:

(a) $N \subseteq M$ is distributive and complemented (there exists H such that $M = N \oplus H$).
(b) N and H have no isomorphic simple subfactors.
(c) $\mathrm{Ann}(n) + \mathrm{Ann}(h) = R$ for any $n \in N$ and any $h \in H$.

We call a direct sum decomposition $M = N \oplus H$, of M, satisfying these equivalent properties, a lattice decomposition of M.

Case of modules II. Endomorphisms

It is well known that if $N \subseteq^{\oplus} M$, there exists an idempotent $e \in \mathrm{End}({}_R M)$ such that $e(M) = N$.

The problem is to characterize e to be N distributive.

If $M = R$, a sufficient condition is that $e \in R = \mathrm{End}({}_R R)$ is central idempotent. In this case the decomposition is $R = Re \oplus R(1-e)$.

If $M \neq R$, this condition is not sufficient. Indeed, in the general case we obtain:
For any submodule $N \subseteq^{\oplus} M$, with idempotent endomorphism $e \in \mathrm{End}({}_R M)$, the following are equivalent

(a) $N = e(M)$ is distributive and complemented.
(b) $e \in \mathrm{End}({}_R M)$ is central idempotent and $e(X) \subseteq X$ for every submodule $X \subseteq M$ (we can say that e is fully invariant).

Application to categories

If M is a left R–module and $M = N \oplus H$ a direct sum decomposition, not necessarily the category $\sigma[M]$ decompose as $\sigma[N] \times \sigma[H]$. But, for lattice decomposition the following are equivalent:

(a) $M = N \oplus H$ is a lattice decomposition.
(b) $\sigma[M] \cong \sigma[N] \times \sigma[H]$.

This decomposition can be extended to any Grothendieck category, even if it has no simple objects.

Application to commutative algebra

Let A be a commutative ring and M be an A–module. If $M = N_1 \oplus N_2$ is a lattice decomposition, then there is a partition of the support of M:

$$\operatorname{Supp}(M) = \operatorname{Supp}(N_1) \stackrel{\bullet}{\cup} \operatorname{Supp}(N_2),$$

and each $\operatorname{Supp}(N_i)$ is closed under *specializations*

$$\text{If } \mathfrak{p} \subseteq \mathfrak{q} \text{ and } \mathfrak{p} \in \operatorname{Supp}(N_i) \text{ then } \mathfrak{q} \in \operatorname{Supp}(N_i).$$

and closed under *generalizations*

$$\text{If } \mathfrak{p} \subseteq \mathfrak{q} \text{ and } \mathfrak{q} \in \operatorname{Supp}(N_i) \text{ then } \mathfrak{p} \in \operatorname{Supp}(N_i).$$

Indeed, we have a characterization of lattice decompositions. The following statements are equivalent:

(a) M has a lattice composition.

(b) $\operatorname{Supp}(M) = C_1 \stackrel{\bullet}{\cup} C_2$, being C_i closed subsets.

Application to commutative algebra II

The behaviour of lattice decomposition under certain constructions is also of interest. Let us show a list of cases:

(1) Lattice decomposition is preserves under localizations; this is because for any prime ideal $\mathfrak{p}$ we have thar either $(N_1)_{\mathfrak{p}} = 0$ or $(N_2)_{\mathfrak{p}} = 0$.
(2) If $A \longrightarrow B$ is a ring map and ${}_BN$ a B–module, and ${}_AN$ has a lattice decomposition, then ${}_BN$ has a lattice decomposition. On the other hand, if ${}_BN$ has a lattice decomposition, not necessarily ${}_AN$ has one.
(3) If $A \longrightarrow B$ is an integral extension and ${}_BN$ has a lattice decomposition, then ${}_AN$ has one.
(4) If $A \longrightarrow B$ is (faithfully) flat and ${}_AM$ is a has a lattice decomposition, then $B \otimes_A M$ has a lattice decomposition.

2 Gradual modules

Let P be a poset, with minimum element 0; it is **directed** if for any $a, b \in P$ there exists $c \in P$ such that $a \leq c$ and $b \leq c$.
We build a category, $\mathcal{P}$ whose objects are the elements of P. For any $a, b \in P$ we define

$$\operatorname{Hom}_{\mathcal{P}}(a, b) = \begin{cases} \{0_{a,b}, f_{a,b}\}, & \text{if } a \leq b, \\ \{0_{a,b}\}, & \text{otherwise,} \end{cases}$$

with composition and addition given, for any $a, b, c \in P$, whenever $a \leq b \leq c$, by the rules:

$$\begin{array}{llll} 0_{b,c}0_{a,b} = 0_{a,c} & 0_{b,c}f_{a,b} = 0_{a,c}; & 0_{a,b} + 0_{a,b} = 0_{a,b} & 0_{a,b} + f_{a,b} = 0_{a,b}; \\ f_{b,c}0_{a,b} = 0_{a,c} & f_{b,c}f_{a,b} = f_{a,c}; & f_{a,b} + 0_{a,b} = 0_{a,b} & f_{a,b} + f_{a,b} = f_{a,b}. \end{array}$$

Let A be a commutative ring, it is possible to modify the above category $\mathcal{P}$ to get a new preadditive A–category, also denoted by $\mathcal{P}$, in defining

$$\mathrm{Hom}_{\mathcal{P}}(a,b) = \begin{cases} \{nf_{a,b} \mid \ n \in A\} = Af_{a,b}, & \text{if } a \leq b \\ \{0_{a,b}\}, & \text{otherwise,} \end{cases}$$

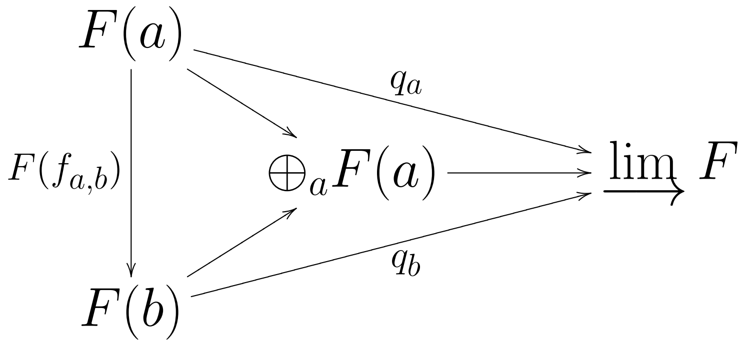
identifying $0_{a,b}$ with $0f_{a,b}$, and $n0_{a,b}$, for any $n \in A$, with addition defined following the addition in A, and composition using the former composition rules.

$\mathcal{P}$ is a preadditive A–category.

Let $F : \mathcal{P} \longrightarrow A\text{–}\mathbf{Mod}$ be an A–additive functor, a left $\mathcal{P}$–module, and consider the family $\{F(a) \mid\ a \in P\}$, and, for any $a, b \in P$ the map $F(f_{a,b}) : F(a) \longrightarrow F(b)$, whenever it exists; in this case we have a directed system of A–modules:

$$(\{F(a) \mid\ a \in P\}, \{F(f_{a,b}) \mid\ a \leq b\}).$$

The existence of the direct limits in $A\text{–}\mathbf{Mod}$ is assured, hence we have an A–module: $\varinjlim F$, and homomorphisms, say $q_a : F(a) \longrightarrow \varinjlim F$, such that the following diagram commutes, for every pair $a \leq b$.



(1)

$f_{a,b}$, whenever $a \leq b$, is an epimorphism and a monomorphism in $\mathcal{P}$.

Let $x \in P$, if we consider the $\mathcal{P}$–module $\mathrm{Hom}_{\mathcal{P}}(x,-)$, we have a module map $(f_{a,b})_* : \mathrm{Hom}_{\mathcal{P}}(x,a) \longrightarrow \mathrm{Hom}_{\mathcal{P}}(x,b)$ which is a monomorphism. In general, $(f_{a,b})_*$ is not an epimorphism as if $a \leq b$ and $0_{x,b} \neq f \in \mathrm{Hom}_{\mathcal{P}}(x,b)$, then $x \leq b$, but it may be $x \not\leq a$, hence $\mathrm{Hom}_{\mathcal{P}}(x,a) = \{0_{x,a}\}$. Also we consider the right $\mathcal{P}$–module $\mathrm{Hom}_{\mathcal{P}}(-,x)$.

In (1), taking $F = \mathrm{Hom}_{\mathcal{P}}(x,-)$, every map $F(f_{a,b})$ is a monomorphism. Hence each map q_a is a monomorphism, i.e., each $\mathrm{Hom}_{\mathcal{P}}(x,a)$ is a submodule of $\varinjlim \mathrm{Hom}_{\mathcal{P}}(x,-)$.

The construction of $\mathrm{Hom}_{\mathcal{P}}(x,-)$ implies that we may identify $\mathrm{Hom}_{\mathcal{P}}(x,a)$ and $Af_{x,a}$, both of them to be isomorphic to A, as A–modules. Otherwise, if $f \in \mathrm{Hom}_{\mathcal{P}}(x,a)$, there exists $n \in A$ such that $f = nf_{x,a}$. Hence, if $x \leq a$, then $(f_{x,a})_* : \mathrm{Hom}_{\mathcal{P}}(x,x) \longrightarrow \mathrm{Hom}_{\mathcal{P}}(x,a)$, and $f = nf_{x,a} = nf_{x,a}f_{x,x} = ff_{x,x} = f \cdot f_{x,x}$. Hence, $f_{x,x}$ generates $\mathrm{Hom}_{\mathcal{P}}(x,-)$.

Each $\mathrm{Hom}_{\mathcal{P}}(x,-)$ is a cyclic $\mathcal{P}$–module with generator $f_{x,x}$.

In the category $\mathcal{P}-\mathbf{Mod}$ we collect in a class all $\mathcal{P}$–modules satisfying the property that each map q_α is a monomorphism. Let F be a $\mathcal{P}$–module, we say F is **torsionfree** if $F(f_{a,b})$ is a monomorphism for every $a \leq b$, and denote by $\mathcal{J}$ the class of all torsionfree $\mathcal{P}$–modules.

The class $\mathcal{J}$ satisfies the following properties:

(1) It is closed under monomorphisms.
(2) It is closed under direct sums and direct products.
(3) It is closed under group–extension.

In particular, the class $\mathcal{J}$ is the torsionfree class of a torsion theory in $\mathcal{P}-\mathbf{Mod}$.

To find this torsion theory, for any $\mathcal{P}$–module, F, and any $a \in P$, we define

$$\eta(F)(a) = \{u \in F(a) \mid \text{ exists } b \in P, a \leq b, \text{ such that } F(f_{a,b})(u) = 0\}.$$

$\eta(F)$ is a submodule of F, and $F/\eta(F)$ is torsionfree.

A $\mathcal{P}$–module F such that $F = \eta(F)$ is called a **torsion** $\mathcal{P}$–module. We may characterize the $\mathcal{P}$–modules which are torsion:

A $\mathcal{P}$–module F is torsion ($F = \eta(F)$) if, and only if, $\varinjlim F = 0$.

The associated Gabriel filter is

$$\mathcal{L}(a) = \{\mathfrak{a} \subseteq \mathrm{Hom}_{\mathcal{P}}(a, -) \mid \varinjlim \mathrm{Hom}(a, -) = \varinjlim \mathfrak{a}\}.$$

Let $F \in \mathcal{J}$ be a torsionfree $\mathcal{P}$–module, for any $a \in P$ we define

$$\begin{array}{l} F^d(0) = F(0), \\ F^d(a) = \sum\{F(b) \mid \ b < a\}, \ \text{if } a \neq 0, \end{array}$$

where this sum is in $\varinjlim F$.

Let F be a torsionfree $\mathcal{P}$–module, then F^d defines a functor from $\mathcal{P}$ to A–**Mod**, hence a $\mathcal{P}$–module, and a submodule of F which is also torsionfree.

This means that the operator $d : \mathcal{J} \longrightarrow \mathcal{J}$, defined by $d(F) = F^d$, is an **interior operator**. Indeed, it satisfies the statements in the following Lemma.

(1) $d(F) \subseteq F$ for any $F \in \mathcal{J}$.
(2) $d(F_1) \subseteq d(F_2)$ whenever $F_1 \subseteq F_2$, for any $F_1, F_2 \in \mathcal{J}$.
(3) $d(F) = dd(F)$ for any $F \in \mathcal{J}$.

A torsionfree $\mathcal{P}$–module is d–**open** if $d(F) = F$.

Let us show some arithmetical properties of this interior operator, with respect to submodules.

(1) Let $\{F_i \mid i \in I\}$ be a family of torsionfree submodules of a $\mathcal{P}$–module F, then

$$\left(\sum_i F_i\right)^d = \sum_i F_i^d.$$

As a submodule of F^d. Thus, the class of d–open submodules is closed under sums.

(2) Let $F_1, F_2 \subseteq F$ be torsionfree submodules of a $\mathcal{P}$–module F, then

$$(F_1 \cap F_2)^d = F_1^d \cap F_2^d.$$

Thus, the class of d–open submodules is closed under finite intersections.

(3) Let $\mathfrak{a}$ be a torsionfree left ideal, and $G \subseteq F$ be a submodule of a torsionfree $\mathcal{P}$–module F, then

$$(\mathfrak{a}G)^d = \mathfrak{a}^d G^d.$$

Thus, the class of d–open left ideals is closed under products.

Let A be a commutative, a fuzzy subset μ is a **fuzzy ideal** if for any $x, y \in A$ we have:

(1) $\mu(x-y) \geq \min\{\mu(x), \mu(y)\}$,
(2) $\mu(xy) \geq \max\{\mu(x), \mu(y)\}$ and
(3) $\mu(0) \neq 0$, to avoid the trivial case.

If μ is a fuzzy ideal, then $\mu(0) \geq \mu(x)$ for any $x \in A$.

For any $\alpha \in [0,1]$, the α–**level** and **strong** α–**level** of a fuzzy ideal μ are defined as:

$$\mu_\alpha = \{x \in A \mid \ \mu(x) \geq \alpha\},$$
$$\widetilde{\mu}_\alpha = \{x \in A \mid \ \mu(x) > \alpha\}.$$

Observe that $\mu_0 = A$; for that reason we shall use α–levels with $\alpha \in (0,1]$.

If μ is a fuzzy ideal, if, and only if, μ_α and $\widetilde{\mu}_\alpha$ are ideals for every $0 \leq \alpha \leq \mu(0)$.

Let μ be a fuzzy ideal of a ring A; if $\mu(x) = \mu(y) = \mu(0)$, then $\mu(x - y) = \mu(0)$.

The problem of working with algebraic operations of fuzzy ideals is hard; if μ_1 and μ_2 are fuzzy ideals, then $\mu_1 + \mu_2$ non–necessarily coincides with the smallest fuzzy ideal containing μ_1 and μ_2; one condition in order to have this property is that $\mu_1(0) = \mu_2(0)$.

A similar problem arise when associating a right $\mathcal{P}$–module to a fuzzy ideal μ. The natural candidate is $\sigma(\mu)$, defined $\sigma(\mu)(\alpha) = \mu_\alpha = \{x \in A \mid \mu(x) \geq \alpha\}$, the α–**level** of μ, which is empty if $\alpha > \mu(0)$.

This second problem can be easily solved if we put $\sigma(\mu)(\alpha) = \{0\}$ whenever $\alpha > \mu(0)$, and this means that a plethora of fuzzy ideals μ have associated the same gradual right ideal: exactly those which coincides in $A \setminus \{0\}$. To organize all fuzzy ideals we may define an equivalence relation $\sim$ on fuzzy ideals by $\mu_1 \sim \mu_2$ if $\mu_1(x) = \mu_2(x)$ for any $0 \neq x \in A$.

Observe that in the equivalence class $[\mu]$ of μ there exists exactly one element, that attending to μ is denoted by μ^0, such that $\mu^0(0) = 1$, i.e., $\mu^0(x) = \begin{cases} \mu(x), & \text{if } x \neq 0, \\ 1, & \text{if } x = 0. \end{cases}$

Let μ be a fuzzy ideal of a ring A, then μ^0 is a fuzzy ideal.

As a consequence we may define a new sum operation on fuzzy ideals using equivalence classes: $[\mu_1] + [\mu_2] = [\mu_1^0 + \mu_2^0]$. Be careful, as the map $(-)^0$ is not necessarily a homomorphism with respect to the sum of fuzzy ideals. If necessary, either we avoid the use of parenthesis, or we adorne the sum symbol, as $[+]$, to indicate we are working with equivalence classes. For two fuzzy ideals μ_1 and μ_2 simply we write

$$([\mu_1] + [\mu_2])(x) = (\mu_1[+]\mu_2)(x) = \mathrm{Sup}\{\mu_1^0(y) \wedge \mu_2^0(z) \mid \ y + z = x\}.$$

In this case, associated to every class $[\mu]$, there exists a right $\mathcal{P}$–module $\sigma(\mu)$ which is a submodule of A, the constant right $\mathcal{P}$–module equal to A, which is identify with the contravariant functor $\mathrm{Hom}_{\mathcal{P}}(-, 1)$.

The maps $\mu \mapsto \sigma(\mu)$ and $\sigma \mapsto \mu(\sigma)$ establish a bijective correspondence between

(i) Equivalence classes of fuzzy ideals and

(ii) Decreasing gradual right ideals (open right ideals in the general theory).

But this correspondence doesn't respect the arithmetical operations. Thus we consider strong α–levels and obtain:

The maps $\mu \mapsto \widetilde{\sigma}(\mu)$ and $\sigma \mapsto \widetilde{\mu}(\sigma)$ establish a bijective correspondence, that maintain the sum and intersections, between

(i) Equivalence classes of fuzzy ideals and

(ii) Decreasing gradual right ideals (open right ideals in the general theory).

This theory can be extended to modules, providing a categorical framework for studying fuzzy modules.