New Hopf algebras arising from the generalized lifting method

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Based on joint work with D. Bagio, J. M. Jury Giraldi and O. Marquez.

[GJG] G. A. García and J. M. Jury Giraldi, On Hopf algebras over quantum subgroups. *J. Pure Appl. Algebra*, Volume **223** (2019), Issue 2, 738–768.

[BGJM] D. Bagio, G. A. García, J. M. Jury Giraldi and O. Marquez, On Hopf algebras over duals of Radford algebras. In preparation.

Let $\Bbbk$ be an algebraically closed field of characteristic zero and let H be a Hopf algebra over $\Bbbk$.

As a coalgebra, H has a canonical coalgebra filtration, the *coradical filtration* $\{H_n\}_{n\geq 0}$:

- $H_0 \subseteq H_1 \subseteq \cdots \subseteq H_n \subseteq \cdots$
- $\bigcup_{n\geq 0} H_n = H$,
- $\Delta(H_n) \subseteq \sum_{i=0}^{n} H_i \otimes H_{n-i}$.

$H_0 =$ coradical of $H =$ sum of all simple subcoalgebras.

$H_n = \bigwedge^{n+1} H_0 = H_{n-1} \wedge H_0$.

$H_n = \{h \in H : \ \Delta(h) \in H \otimes H_{n-1} + H_0 \otimes H\}$.

One has that $H_0 = \mathrm{Jac}(H^*)^{\perp}$ and $H_n = (\mathrm{Jac}(H^*)^{n+1})^{\perp}$.

If H_0 is a **Hopf** subalgebra, then the filtration is a Hopf algebra filtration and

$$\operatorname{gr} H = \bigoplus H_n/H_{n-1}, \qquad \text{with } H_{-1} = 0$$

is a Hopf algebra.

Take the homogeneous projection

$$\pi : \operatorname{gr} H \twoheadrightarrow H_0.$$

It has a Hopf algebra section (the inclusion) and

$\operatorname{gr} H \simeq R \# H_0$ **Majid-Radford product or bosonization**

here $R = (\operatorname{gr} H)^{\mathrm{co}\,\pi}$ a braided graded Hopf algebra in ${}^{H_0}_{H_0}\mathcal{YD}$.

H is called a *lifting* of R over H_0.

Let $V = P(R) = \{r \in R : \ \Delta(r) = r \otimes 1 + 1 \otimes r\}$ be the space of primitive elements.

The subalgebra $\mathfrak{B}(V)$ of R generated by V is called the *Nichols algebra* of V:

- $\mathfrak{B}(V)$ is graded with $\mathfrak{B}(V)(0) = \Bbbk$ and $\mathfrak{B}(V)(1) = V$.
- $\mathfrak{B}(V)(1) = P(\mathfrak{B}(V))$.
- $\mathfrak{B}(V)$ is generated by V.

Rmk: It is possible to define $\mathfrak{B}(V)$ in terms of the braided vector space (V, c): $\mathfrak{B}(V) = T(V)/J$, with J the largest two-sided ideal and coideal $J \subseteq \oplus_{n \geq 2} V^n$.

The Lifting Method for fin-dim. Hopf algebras [Andruskiewitsch-Schneider]

Let A be a finite-dimensional cosemisimple Hopf algebra.

(*a*) Determine $V \in {}^{A}_{A}\mathcal{YD}$ such that $\mathfrak{B}(V)$ is finite-dimensional.

(*b*) For such V, compute all L s.t. $\operatorname{gr} L \simeq \mathfrak{B}(V)\#A$.

(*c*) Prove that for all H such that $H_0 = A$, then $\operatorname{gr} H \simeq \mathfrak{B}(V)\#A$. (generation in degree one)

Assume $A = \Bbbk\Gamma$ group algebra over a finite group $\rightsquigarrow$ pointed Hopf algebras

- Classification obtained for Γ abelian.
- Few examples for Γ non-abelian: *e.g.* $\mathbb{S}_3$, $\mathbb{S}_4$, $\mathbb{D}_{4t}$, $\mathbb{Z}_r \ltimes \mathbb{Z}_s$.

Conjecture

Any finite-dimensional pointed Hopf algebra H s.t. $H_0 \simeq \Bbbk\Gamma$, with Γ finite non-abelian simple group is trivial, *i.e.* $H \simeq \Bbbk\Gamma$.

Verified for $\mathbb{A}_n$ with $n \geq 5$, almost all sporadic groups, Suzuki-Ree groups and infinite families of finite simple groups of Lie type

What if H_0 is not a Hopf subalgebra?

[Andruskiewitsch-Cuadra]: replace the coradical filtration by a more general but adequate one $\rightsquigarrow$ the **standard** filtration $\{H_{[n]}\}_{n\geq 0}$

- the subalgebra $H_{[0]}$ of H generated by H_0, called the *Hopf coradical*,
- $H_{[n]} = \bigwedge^{n+1} H_{[0]}$.

It holds: If S is bijective then $H_{[0]}$ is a Hopf subalgebra of H, $H_n \subseteq H_{[n]}$ and $\{H_{[n]}\}_{n\geq 0}$ is a Hopf algebra filtration of H.

In particular,

$$\operatorname{gr} H = \bigoplus_{n\geq 0} H_{[n]}/H_{[n-1]} \qquad \text{is a Hopf algebra}$$

If H_0 is a Hopf subalgebra, then $H_{[0]} = H_0$ and the coradical filtration coincides with the standard one.

The Generalized Lifting Method for fin-dim. Hopf algebras [Andruskiewitsch-Cuadra]

Let A be a finite-dimensional generated by a **cosemisimple coalgebra**.

(a) Determine $V \in {}^{A}_{A}\mathcal{YD}$ such that $\mathfrak{B}(V)$ is finite-dimensional.

(b) For such V, compute all L s.t. $\operatorname{gr} L \simeq \mathfrak{B}(V)\#A$.

(c) Prove that for all H such that $H_{[0]} = A$, then $\operatorname{gr} H \simeq \mathfrak{B}(V)\#A$. (generation in degree one w.r.t. the **standard filtration**)

First goal: Construct new Hopf algebras based on this method.

First obstruction: find Hopf algebras generated by their coradicals.

Source of examples: quotients of quantum function algebras:

Let ξ be a primitive 4-th root of 1 and let $\mathcal{K}$ be generated by a, b, c, d satisfying

$$\begin{array}{lllll} ab = \xi ba, & ac = \xi ca, & 0 = cb = bc, & cd = \xi dc, & bd = \xi db, \\ ad = da, & ad = 1, & 0 = b^2 = c^2, & a^2 c = b, & a^4 = 1. \end{array}$$

The coalgebra structure and its antipode are determined by

$$\begin{array}{ll} \Delta(a) = a \otimes a + b \otimes c, & \Delta(b) = a \otimes b + b \otimes d, \\ \Delta(c) = c \otimes a + d \otimes c, & \Delta(d) = c \otimes b + d \otimes d, \\ \varepsilon(a) = 1, \quad \varepsilon(b) = 0, & \varepsilon(c) = 0, \quad \varepsilon(d) = 1 \\ S(a) = d, \quad S(b) = \xi b, & S(c) = -\xi c, \quad S(d) = a. \end{array}$$

$\mathcal{K}$ is an 8-dimensional Hopf algebra, it is a quotient of $\mathcal{O}_q(\mathbf{SL}_2)$, and $\mathcal{K}^*$ is a pointed Hopf algebra $\rightsquigarrow$ *basic* Hopf algebra.

$\mathcal{K}^* = R_{2,2}$ was first introduced by Radford.

Duals of general Radford algebras $R_{n,m}$ satisfy this property.

Prop-Def (Andruskiewitsch-Cuadra-Etingof)

Let $\xi \in \mathbb{G}'_{nm}$. $\mathcal{K}_{n,m} = R^*_{n,m}$ is generated by U, X and A satisfying

$$U^n = 1, \qquad X^n = 0, \qquad A^m = U,$$
$$UX = \omega XU, \qquad UA = AU, \qquad AX = \xi XA.$$

As coalgebra $U \in G(\mathcal{K}_{n,m})$, $X \in \mathcal{P}_{1,U}(\mathcal{K}_{n,m})$ and

$$\Delta(A) = A \otimes A + \sum_{k=1}^{n-1} \gamma_{n,k} X^{n-k} U^k A \otimes X^k A$$

where $\gamma_{n,k} = \frac{1-\xi^n}{(k)!_\omega (n-k)!_\omega}$.

Write $\mathcal{K} = \mathcal{K}_{n,m}$.

Step (a):
To describe $V \in {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$, we use the equivalence ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD} \simeq {}_{D(\mathcal{K}^{\mathrm{cop}})}\mathcal{M}$.

$D(\mathcal{K}^{\mathrm{cop}}) = D$ is a non-semisimple Hopf algebra of tame representation type $\rightsquigarrow$ we describe the simple modules, their projective covers and some indecomposable modules.

For $0 \leq i, j \leq nm - 1$, let $r_{ij} \in \mathbb{N}$ such that $1 \leq r_{ij} \leq n$ and

$$r_{ij} = \begin{cases} i + \frac{j}{m} + 1 \mod n & \text{if} \quad m \mid j, \\ n & \text{if} \quad m \nmid j. \end{cases}$$

Definition

Let $0 \le i, j < nm$ and write $r = r_{i,j}$. Let $V_{i,j}$ be the $\mathbb{C}$-vector space with basis $B = \{v_0, \cdots, v_{r-1}\}$ and D-action given by

$$A \cdot v_k = \xi^{i-k} v_k \qquad g \cdot v_k = \xi^{j-km} v_k \qquad \forall\ 0 \le k \le r-1,$$

$$x \cdot v_k = \begin{cases} v_{k+1} & \text{if} \quad 0 \le k < r-1, \\ (1-\xi^{jn}) v_0 & \text{if} \quad k = r-1, \end{cases}$$

$$X \cdot v_k = \begin{cases} 0 & \text{if} \quad k = 0, \\ c_k v_{k-1} & \text{if} \quad 0 < k \le r-1, \end{cases}$$

where

$$c_k = (k)_\omega\, \omega^{-k} (\xi^j \omega^{-k+1+i} - 1), \qquad \forall\ 1 \le k \le r-1. \tag{1}$$

Theorem (Bagio, G, Jury Giraldi, Marquez)

$\{V_{i,j}\}_{1\leq i,j<nm}$ *is a set of pairwise non-isomorphic simple* D*-modules.*

The case $n = 2 = m$

Theorem (G-Jury Giraldi)

Let $M \in {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$ *be a finite-dimensional non-simple indecomposable module. Then* $\mathfrak{B}(M)$ *is infinite-dimensional.*

Theorem (G-Jury Giraldi, Xiong, Andruskiewitsch-Angiono)

Let $\mathfrak{B}(V)$ *be a finite-dimensional Nichols algebra over an object* V *in* ${}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD}$*. Then* V *is semisimple and isomorphic either to* $\Bbbk_{\chi^j} = V_{j,2}$, $V_{1,j}$, $V_{2,j}$, $\bigoplus_{\ell=1 \text{ or } 3} \Bbbk_{\chi^\ell}$, $V_{1,j} \oplus \Bbbk_{\chi}$, $V_{2,j} \oplus \Bbbk_{\chi^3}$, $V_{1,1} \oplus V_{1,3}$, $V_{2,1} \oplus V_{2,3}$ *with* $j = 1, 3$.

- $\mathfrak{B}(\bigoplus_{i=1}^{n} \Bbbk_{\chi^{\ell_i}}) = \bigwedge_{i=1}^{n} \Bbbk_{\chi^{\ell_i}}$, $\dim \mathfrak{B}(\bigoplus_{i=1}^{n} \Bbbk_{\chi^{\ell_i}}) = 2^n$.
- $\mathfrak{B}(V_{1,j}) = \Bbbk\langle x, y : \; x^2 + 2\xi y^2 = 0, xy + yx = 0, x^4 = 0\rangle$, $\dim \mathfrak{B}(V_{1,j}) = 8$. The braiding is **not** diagonal $\rightsquigarrow$ new example!
- $\mathfrak{B}(V_{1,j} \oplus \Bbbk_\chi) = \Bbbk\langle x, y, z \rangle / J$, with J generated by:

$$x^2 + 2\xi y^2 = 0, \quad xy + yx = 0, \quad x^4 = 0, \quad z^2 = 0,$$
$$zx^2 + (1 - \xi^j)xzx - \xi^j x^2 z = 0,$$
$$\xi^j xyz - \xi^j xzy + yzx + zxy = 0,$$
$$\frac{1}{2}\xi(1 + \xi^{-j})(xz)^2(yz)^2 + (yz)^4 + (zy)^4 = 0.$$

$\dim \mathfrak{B}(V_{1,j} \oplus \Bbbk_\chi) = 128$.

Theorem (G-Jury Giraldi, Xiong, Andruskiewitsch-Angiono)

Let H be a finite-dimensional Hopf algebra over $\mathcal{K}$. Then H is isomorphic either to

(i) $(\bigwedge_{i=1}^{n} \Bbbk_{\chi^{\ell_i}})\#\mathcal{K}$ *with* $\ell_i = 1, 3$;

(ii) $\mathfrak{B}(V_{2,j})\#\mathcal{K}$ *for* $j = 1, 3$;

(iii) $\mathfrak{B}(V_{2,j} \oplus \Bbbk_{\chi^3})\#\mathcal{K}$;

(iv) $\mathfrak{B}(V_{2,1} \oplus V_{2,3})\#\mathcal{K}$

(v) $A_{1,j}(\mu)$ *for* $j = 1, 3$ *and some* $\mu \in \Bbbk$;

(vi) $A_{1,j,1}(\mu, \nu)$ *for* $j = 1, 3$ *and some* $\mu, \nu \in \Bbbk$.

(vii) $A_{1,1,1,3}(\mu, \nu)$ *for* $j = 1, 3$ *and some* $\mu, \nu \in \Bbbk$.

Let $j \in \{1,3\}$ and $\mu \in \Bbbk$. The algebra $A_{1,j}$ is generated by a, b, x, y satisfying (change $A = a$ and $X = b$):

$$a^4 = 1, \quad b^2 = 0, \quad ba = \xi ab, \quad , ax = \xi xa, \quad bx = \xi xb,$$
$$ay + ya = \xi^3 xba^2, \quad by + yb = xa^3,$$
$$x^4 = 0, \quad x^2 + 2\xi y^2 = \mu(1 - a^2), \quad xy + yx = \mu\xi^3 ba^3.$$

For the coproduct, one has that

$$\Delta(a) = a \otimes a + \xi^{-1} b \otimes ba^2,$$
$$\Delta(b) = b \otimes a^3 + a \otimes b,$$
$$\Delta(x) = x \otimes 1 + a^{-j} \otimes x - (1 + \xi^j) ba^{-1-j} \otimes y,$$
$$\Delta(y) = y \otimes 1 + a^{2-j} \otimes y + \frac{1}{2}\xi(1 - \xi^j) ba^{1-j} \otimes x.$$

For Nichols algebras in the general case $\rightsquigarrow$ use techniques of Andruskiewitsch-Angiono to complete Step (a).

Idea:

- $\mathcal{K}^* = R_{n,m}$ is pointed, *i.e.* $\mathcal{K}$ is basic.
- $R_{n,m} \simeq (T_{n,m})_\sigma$, the generalized Taft algebra

$$T_{n,m} = \Bbbk\langle g, x : \; x^n = 0, g^{nm} = 1, gx = \xi^m xg\rangle$$
$$\simeq \big(\Bbbk[x]/(x^n)\big)\#\Bbbk C_{nm} = \mathfrak{B}(V)\#\Bbbk C_{nm}, \quad V = \Bbbk x.$$

Also, the 2-cocycle σ is known!

- Use the composition of braided monoidal equivalences

$$F : \; {}_D\mathcal{M} \xrightarrow{F_1} {}^{\mathcal{K}}_{\mathcal{K}}\mathcal{YD} \xrightarrow{F_2} {}^{R_{n,m}}_{R_{n.m}}\mathcal{YD} \xrightarrow{F_3} {}^{T_{n,m}}_{T_{n.m}}\mathcal{YD}$$

Let $\lambda_{i,j}$ denote a simple object of ${}^{C_{nm}}_{C_{nm}}\mathcal{YD}$.

Let $L(\lambda_{i,j})$ be the corresponding simple object in ${}^{T_{nm}}_{T_{nm}}\mathcal{YD}$.

Fact: It holds that $F(V_{i,j}) = L(\lambda_{-i,-j})$ for all $0 \leq i, j < nm$.

Theorem (Andruskiewitsch-Angiono)

Let $V_{i,j}$ be a D-simple module. Then $\dim \mathfrak{B}(V_{i,j}) < \infty$ if and only if $\dim \mathfrak{B}(V \oplus \lambda_{-i,-j}) < \infty$.

Remark: $V \oplus \lambda_{-i,-j}$ is a braided vector space of diagonal type $\rightsquigarrow$ we know exaclty when $\dim \mathfrak{B}(V \oplus \lambda_{-i,-j}) < \infty$.

Difficult step: Find the presentation of those $\mathfrak{B}(V_{i,j})$ such that $\dim \mathfrak{B}(V_{i,j}) < \infty$.

We have infinite families!!