Pretorsion theories in arbitrary categories

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Dedicated to Blas,. . .

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A. Facchini and C. Finocchiaro, *Pretorsion theories, stable category and preordered sets,* submitted for publication, arXiv:1902.06694, 2019.

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A. Facchini and L. Heidari Zadeh, *An extension of properties of symmetric group to monoids and a pretorsion theory in the category of mappings,* to appear, arXiv:1902.05507, 2019.

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(5) For $n \geq 2$, S_n is the semidirect product of A_n and any subgroup of S_n generated by a transposition.

Every permutation is a product of disjoint cycles

Given any mapping $f: X \to X$, it is possible to associate to f a directed graph $G_f^d = (X, E_f^d)$ (the *graph associated to the function* f), having X as a set of vertices and $E_f^d := \{ (i, f(i)) \mid i \in X \}$ as a set of arrows. Hence G_f^d has n vertices and n arrows, one arrow from i to $f(i)$ for every $i \in X$. In the directed graph G_f^d, every vertex has outdegree 1.

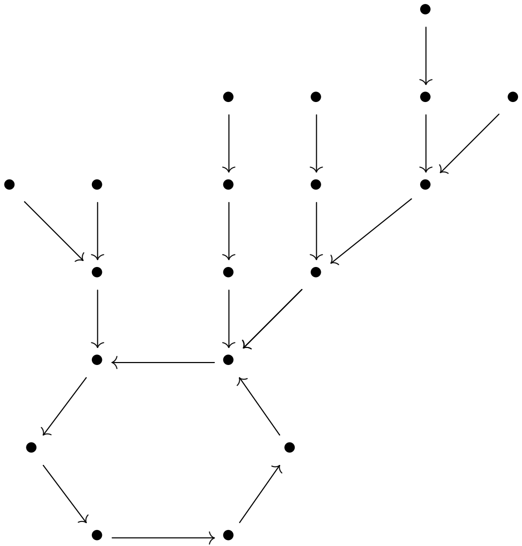
Every permutation is a product of disjoint cycles

If $f\colon X \to X$ is a permutation, every vertex in G_f^d has outdegree 1 and indegree 1. Any finite directed connected graph in which every vertex has outdegree 1 and indegree 1 is a cycle. Therefore the graph G_f^d, disjoint union of its connected components, is a disjoint union of cycles in a unique way. Hence any permutation f is a product of disjoint cycles.

For an arbitrary mapping $f\colon X \to X \ldots$

For any mapping $f\colon X \to X$, we can argue in the same way, but instead of a disjoint union of cycles, we get as G_f^d a disjoint union of forests on cycles:

A forest on a cycle



An arbitrary mapping $f\colon X \to X \ldots$

Any mapping $f\colon X \to X$ consists a lower part (a disjoint union of cycles, i.e., a bijection) and an upper part (a forest).

For a mapping $f\colon X = \{1, 2, 3, \ldots, n\} \to X = \{1, 2, 3, \ldots, n\}$:
(1) f is a bijection if and only if $f^{n!} = \iota_X$.
(2) The graph G_f^d is a forest (i.e., the only cycles on G_f^d are the loops) if and only if $f^n = f^{n+1}$.

The category of mappings $\mathcal{M}$

Let $\mathcal{M}$ be the category whose objects are all pairs (X, f), where $X = \{1, 2, 3, \dots, n\}$ for some $n \geq 1$ and $f \colon X \to X$ is a mapping.

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A morphism $g\colon (X, f) \to (X', f')$ in $\mathcal{M}$ is any mapping $g\colon X \to X'$ for which the diagram

$$\begin{array}{ccc} X & \xrightarrow{g} & X' \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle f'} \\ X & \xrightarrow[g]{} & X' \end{array} \tag{1}$$

commutes.

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A *congruence* on (X, f), in the sense of Universal Algebra, is an equivalence relation $\sim$ on the set X such that, for all $x, y \in X$, $x \sim y$ implies $f(x) \sim f(y)$.

The pretorsion theory $(\mathcal{C}, \mathcal{F})$ on $\mathcal{M}$

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Clearly, an object of $\mathcal{M}$ is an object both in $\mathcal{C}$ and in $\mathcal{F}$ if and only if it is of the form (X, ι_X), where $\iota_X : X \to X$ is the identity mapping.

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Call a morphism $g \colon (X, f) \to (X', f')$ in $\mathcal{M}$ *trival* if it factors through a trivial object. That is, if there exists a trivial object (Y, ι_Y) and morphisms $h \colon (X, f) \to (Y, \iota_Y)$ and $\ell \colon (Y, \iota_Y) \to (X', f')$ in $\mathcal{M}$ such that $g = \ell h$.

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Proposition

If (X, f) and (X', f') are objects of $\mathcal{M}$, where f is a bijection and the graph of f' is a forest, then every morphism $g\colon (X, f) \to (X', f')$ is trivial.

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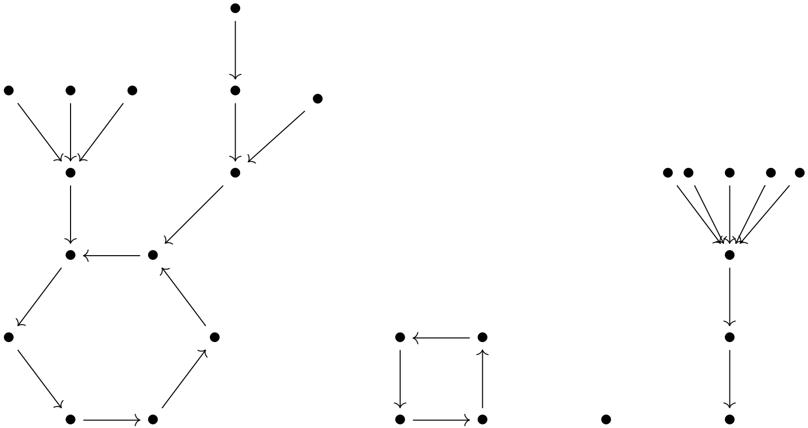
We will see that for every object (X, f) in $\mathcal{M}$ there is a "short exact sequence"

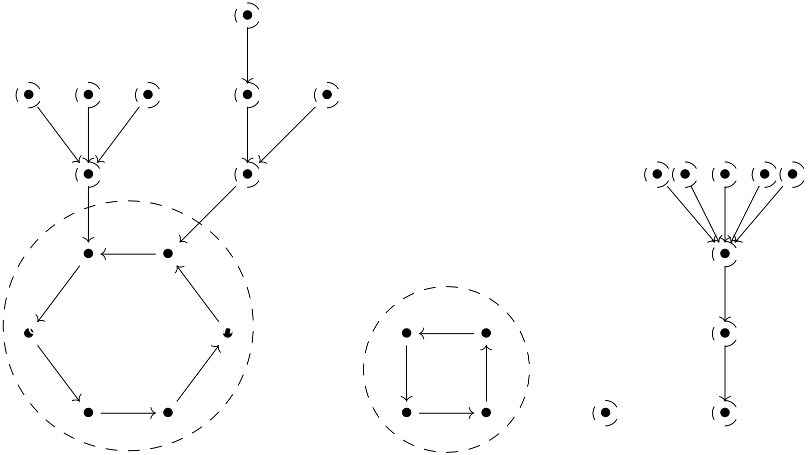
$$(A_0, f|_{A_0}^{A_0}) \overset{\varepsilon}{\hookrightarrow} (X, f) \overset{\pi}{\twoheadrightarrow} (X/{\sim}, \overline{f}) \tag{2}$$

with $(A_0, f|_{A_0}^{A_0}) \in \mathcal{C}$ and $(X/{\sim}, \overline{f}) \in \mathcal{F}$.

An example: an object (X, f) in $\mathcal{M}$



The partition of X modulo $\sim$.



The torsion-free quotient $(X/\sim, \overline{f})$ of $(X.f)$ modulo $\sim$.

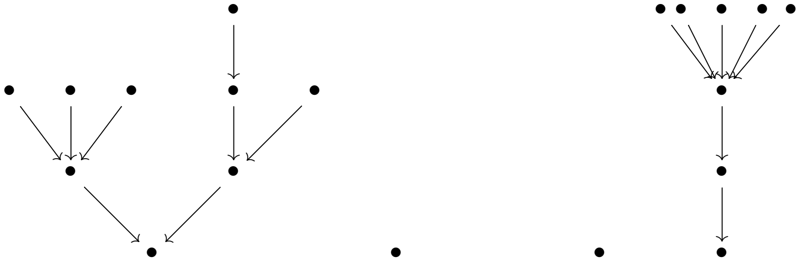


Figure: The quotient set $X/\sim$.

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(1) partial orders (i.e., ρ is also antisymmetric).

(2) equivalence relations (i.e., ρ is also symmetric).

Proposition

Let A be a set. There is a one-to-one correspondence between the set of all preorders ρ on A and the set of all pairs $(\sim, \leq)$, where $\sim$ is an equivalence relation on A and $\leq$ is a partial order on the quotient set $A/\sim$.

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The correspondence associates to every preorder ρ on A the pair $(\simeq_\rho, \leq_\rho)$, where $\simeq_\rho$ is the equivalence relation defined, for every $a, b \in A$, by $a \simeq_\rho b$ if $a\rho b$ and $b\rho a$, and $\leq_\rho$ is the partial order on $A/\simeq_\rho$ defined, for every $a, b \in A$, by $[a]_{\simeq_\rho} \leq [b]_{\simeq_\rho}$ if $a\rho b$.

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Conversely, for any pair $(\sim, \leq)$ with $\sim$ an equivalence relation on A and $\leq$ a partial order on $A/{\sim}$, the corresponding preorder $\rho_{(\sim,\leq)}$ on A is defined, for every $a, b \in A$, by $a\rho_{(\sim,\leq)}b$ if $[a]_\sim \leq [b]_\sim$.

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Equiv: full subcategory of **Preord** whose objects are all preordered sets $(A, \sim)$ with $\sim$ an equivalence relation on A.

Trivial objects, trivial morphisms

Triv:=**Preord** ∩ **Equiv**, full subcategory of **Preord** whose objects are all the objects of the form $(A, =)$, where $=$ denotes the equality relation on A. We will call them the *trivial objects* of **Preord**.

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A morphism $f \colon (A, \rho) \to (A', \rho')$ in **Preord** is *trival* if it factors through a trivial object, that is, if there exist a trivial object $(B, =)$ and morphisms $g \colon (A, \rho) \to (B, =)$ and $h \colon (B, =) \to (A', \rho')$ in **Preord** with $f = hg$.

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1. fk is a trivial morphism.
2. Whenever $\lambda\colon Y \to A$ is a morphism in **Preord** and $f\lambda$ is trivial, then there exists a unique morphism $\lambda'\colon Y \to X$ in **Preord** such that $\lambda = k\lambda'$.

Prekernel of a morphism $f\colon A \to A'$ in **Preord**

For every mapping $f\colon A \to A'$, the equivalence relation $\sim_f$ on A, associated to f, is defined, for every $a, b \in A$, by $a \sim_f b$ if $f(a) = f(b)$.

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Proposition

Let $f\colon (A,\rho) \to (A',\rho')$ be a morphism in **Preord**. *Then a prekernel of f is the morphism $k\colon (A, \rho \cap {\sim_f}) \to (A,\rho)$, where k the identity mapping and $\sim_f$ is the equivalence relation on A associated to f.*

Precokernels

Let $f\colon A \to A'$ be a morphism in **Preord**. A *precokernel* of f is a morphism $p\colon A' \to X$ such that:

1. pf is a trivial map.
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Let $f\colon X \to Y$ and $g\colon Y \to Z$ be morphisms in **Preord**. We say that $X \xrightarrow{f} Y \xrightarrow{g} Z$ is a *short preexact sequence* in **Preord** if f is a prekernel of g and g is a precokernel of f.

A canonical short preexact sequence for every (A, ρ) in **Preord**.

Let A be any non-empty set, let ρ be a preorder on A and let $\simeq_\rho$ be the equivalence relation on A defined by $a \simeq_\rho b$ if $a\rho b$ and $b\rho a$ and $\leq_\rho$ is the partial order on $A/\simeq_\rho$ induced by ρ, then

$$(A, \simeq_\rho) \xrightarrow{\ k\ } (A, \rho) \xrightarrow{\ \pi\ } (A/\simeq_\rho, \leq_\rho)$$

is a short preexact sequence in **Preord** with $(A, \simeq_\rho) \in$ **Equiv** and $(A/\simeq_\rho, \leq_\rho) \in$ **ParOrd**.

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Pretorsion theories

Fix an arbitrary category $\mathcal{C}$ and a non-empty class $\mathcal{Z}$ of objects of $\mathcal{C}$. For every pair A, A' of objects of $\mathcal{C}$, we indicate by $\mathbf{Triv}_{\mathcal{Z}}(A, B)$ the set of all morphisms in $\mathcal{C}$ that factor through an object of $\mathcal{Z}$. We will call these morphisms *$\mathcal{Z}$-trivial*.

Let $f \colon A \to A'$ be a morphism in $\mathcal{C}$. We say that a morphism $\varepsilon \colon X \to A$ in $\mathcal{C}$ is a *$\mathcal{Z}$-prekernel* of f if the following properties are satisfied:

1. $f\varepsilon$ is a $\mathcal{Z}$-trivial morphism.
2. Whenever $\lambda \colon Y \to A$ is a morphism in $\mathcal{C}$ and $f\lambda$ is $\mathcal{Z}$-trivial, then there exists a unique morphism $\lambda' \colon Y \to X$ in $\mathcal{C}$ such that $\lambda = \varepsilon\lambda'$.

Pretorsion theories

Proposition

Let $f\colon A \to A'$ be a morphism in $\mathcal{C}$ and let $\varepsilon\colon X \to A$ be a $\mathcal{Z}$-prekernel for f. Then the following properties hold.

1. *ε is a monomorphism.*
2. *If $\lambda\colon Y \to A$ is any other $\mathcal{Z}$-prekernel of f, then there exists a unique isomorphism $\lambda'\colon Y \to X$ such that $\lambda = \varepsilon\lambda'$.*

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Dually, a *$\mathcal{Z}$-precokernel* of f is a morphism $\eta\colon A' \to X$ such that:

1. ηf is a $\mathcal{Z}$-trivial morphism.
2. Whenever $\mu\colon A' \to Y$ is a morphism and μf is $\mathcal{Z}$-trivial, then there exists a unique morphism $\mu'\colon X \to Y$ with $\mu = \mu'\eta$.

Pretorsion theories

If $\mathcal{C}^{\mathrm{op}}$ is the opposite category of $\mathcal{C}$, the $\mathcal{Z}$-precokernel of a morphism $f\colon A \to A'$ in $\mathcal{C}$ is the $\mathcal{Z}$-prekernel of the morphism $f\colon A' \to A$ in $\mathcal{C}^{\mathrm{op}}$.

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Let $f\colon A \to B$ and $g\colon B \to C$ be morphisms in $\mathcal{C}$. We say that

$$A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$$

is a *short $\mathcal{Z}$-preexact sequence* in $\mathcal{C}$ if f is a $\mathcal{Z}$-prekernel of g and g is a $\mathcal{Z}$-precokernel of f.

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Clearly, if $A \xrightarrow{\ f\ } B \xrightarrow{\ g\ } C$ is a short $\mathcal{Z}$-preexact sequence in $\mathcal{C}$, then
$C \xrightarrow{\ g\ } B \xrightarrow{\ f\ } A$ is a short $\mathcal{Z}$-preexact sequence in $\mathcal{C}^{\mathrm{op}}$.

Pretorsion theories: definition

Let $\mathcal{C}$ be an arbitrary category. A *pretorsion theory* $(\mathcal{T}, \mathcal{F})$ for $\mathcal{C}$ consists of two replete ($=$ closed under isomorphism) full subcategories $\mathcal{T}, \mathcal{F}$ of $\mathcal{C}$, satisfying the following two conditions.

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(1) $\mathrm{Hom}_{\mathcal{C}}(T, F) = \mathbf{Triv}_{\mathcal{Z}}(T, F)$ for every object $T \in \mathcal{T}$, $F \in \mathcal{F}$.

(2) For every object B of $\mathcal{C}$ there is a short $\mathcal{Z}$-preexact sequence

$$A \xrightarrow{f} B \xrightarrow{g} C$$

with $A \in \mathcal{T}$ and $C \in \mathcal{F}$.

Like torsion theories in the abelian case

In the rest of the talk, whenever we will deal with a pretorsion theory $(\mathcal{T}, \mathcal{F})$ for a category $\mathcal{C}$, the symbol $\mathcal{Z}$ will always indicate the intersection $\mathcal{T} \cap \mathcal{F}$.

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Notice that if $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory for a category $\mathcal{C}$, then $(\mathcal{F}, \mathcal{T})$ turns out to be a pretorsion theory in $\mathcal{C}^{\mathsf{op}}$.

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Notice that if $(\mathcal{T}, \mathcal{F})$ is a pretorsion theory for a category $\mathcal{C}$, then $(\mathcal{F}, \mathcal{T})$ turns out to be a pretorsion theory in $\mathcal{C}^{\mathsf{op}}$.

Proposition

Let $(\mathcal{T}, \mathcal{F})$ be a pretorsion theory in a category $\mathcal{C}$, and let X be any object in $\mathcal{C}$.

1. *If $\mathrm{Hom}_{\mathcal{C}}(X, F) = \mathbf{Triv}_{\mathcal{Z}}(X, F)$ for every $F \in \mathcal{F}$, then $X \in \mathcal{T}$.*
2. *If $\mathrm{Hom}_{\mathcal{C}}(T, X) = \mathbf{Triv}_{\mathcal{Z}}(T, X)$ for every $T \in \mathcal{T}$, then $X \in \mathcal{F}$.*

First properties

As a corollary, from Proposition 1.4 we have that given a pretorsion theory $(\mathcal{T}, \mathcal{F})$ in a category $\mathcal{C}$, any two of the three classes $\mathcal{T}, \mathcal{F}, \mathcal{Z}$ determine the third.

First properties

As a corollary, from Proposition 1.4 we have that given a pretorsion theory $(\mathcal{T}, \mathcal{F})$ in a category $\mathcal{C}$, any two of the three classes $\mathcal{T}, \mathcal{F}, \mathcal{Z}$ determine the third.

First of all, we have that the short $\mathcal{Z}$-preexact sequence given in Axiom (2) of the definition of pretorsion theory is uniquely determined, up to isomorphism.

Uniqueness of the short $\mathcal{Z}$-preexact sequence

Proposition

Let $\mathcal{C}$ be a category and let $(\mathcal{T}, \mathcal{F})$ be a pretorsion theory for $\mathcal{C}$. If

$$T \xrightarrow{\varepsilon} A \xrightarrow{\eta} F \quad \text{and} \quad T' \xrightarrow{\varepsilon'} A \xrightarrow{\eta'} F'$$

are $\mathcal{Z}$-preexact sequences, where $T, T' \in \mathcal{T}$ and $F, F' \in \mathcal{F}$, then there exist a unique isomorphism $\alpha\colon T \to T'$ and a unique isomorphism $\sigma\colon F \to F'$ making the diagram

$$\begin{array}{ccccc} T & \xrightarrow{\varepsilon} & A & \xrightarrow{\eta} & F \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle=} & & \downarrow{\scriptstyle\sigma} \\ T' & \xrightarrow{\varepsilon'} & A & \xrightarrow{\eta'} & F' \end{array}$$

commute.

Torsion subobject and torsion-free quotient object are functors

Proposition

Let $(\mathcal{T}, \mathcal{F})$ *be a pretorsion theory for a category* $\mathcal{C}$*.*

Torsion subobject and torsion-free quotient object are functors

Proposition

Let $(\mathcal{T}, \mathcal{F})$ be a pretorsion theory for a category $\mathcal{C}$. Choose, for every $X \in \mathcal{C}$, a short $\mathcal{Z}$-preexact sequence

$$t(X) \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} f(X) \ ,$$

where $t(X) \in \mathcal{T}$ and $f(X) \in \mathcal{F}$.

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Proposition

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$$t(X) \xrightarrow{\varepsilon_X} X \xrightarrow{\eta_X} f(X) \ ,$$

where $t(X) \in \mathcal{T}$ and $f(X) \in \mathcal{F}$. Then the assignments $A \mapsto t(A)$, (resp., $A \mapsto f(A)$) extends to a functor $t \colon \mathcal{C} \to \mathcal{T}$ (resp., $f \colon \mathcal{C} \to \mathcal{F}$).

Torsion subobject and torsion-free quotient object are functors

If, for every $X \in \mathcal{C}$, we chose another short $\mathcal{Z}$-preexact sequence

$$t'(X) \xrightarrow{\lambda_X} X \xrightarrow{\pi_X} f'(X)$$

with $t'(X) \in \mathcal{T}, f'(X) \in \mathcal{F}$, and $t' \colon \mathcal{C} \to \mathcal{T},\ f' \colon \mathcal{C} \to \mathcal{F}$ are the functors corresponding to the new choice, then there is a unique natural isomorphism of functors $t \to t'$ (resp., $f \to f'$).

$\mathcal{T}$ is a coreflective subcategory of $\mathcal{C}$

Theorem

Let $(\mathcal{T}, \mathcal{F})$ be a pretorsion theory for a category $\mathcal{C}$. Then the functor t is a right adjoint of the category embedding $e_{\mathcal{T}} \colon \mathcal{T} \hookrightarrow \mathcal{C}$, so that $\mathcal{T}$ is a coreflective subcategory of $\mathcal{C}$.

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Dually, f is a left adjoint of the embedding $e_{\mathcal{F}} \colon \mathcal{F} \hookrightarrow \mathcal{C}$ and $\mathcal{F}$ is a reflective subcategory of $\mathcal{C}$.

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Cocommutative Hopf algebras

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