Group gradings on matrix algebras

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Let k be a field, and let G be a group.

A G-graded algebra (over k) is a k-algebra A with a decomposition

$$A = \oplus_{g \in G} A_g$$

as a sum of k-subspaces, such that

$$A_g A_h \subset A_{gh}$$

for any $g, h \in G$.

General Problem. *If A is a k-algebra, determine (or even classify) all possible group gradings on A.*

We are interested in the case where A is a structural matrix algebra over k, i.e. a subalgebra of $M_n(k)$ consisting of all matrices with zero entries on certain prescribed positions, and allowing anything on the other positions. For example

$$A = \begin{pmatrix} k & 0 & k & k & k & 0 & k & k \\ 0 & k & 0 & k & 0 & k & k & k \\ k & 0 & k & k & k & 0 & k & k \\ 0 & 0 & 0 & k & 0 & 0 & 0 & 0 \\ k & 0 & k & k & k & 0 & k & k \\ 0 & k & 0 & k & 0 & k & k & k \\ 0 & 0 & 0 & 0 & 0 & 0 & k & k \\ 0 & 0 & 0 & 0 & 0 & 0 & k & k \end{pmatrix}$$

The full matrix algebra $M_n(k)$ and the diagonal algebra k^n are examples of structural matrix algebras.

• Gradings on the full matrix algebra were considered for by Knus in 1969 in his Brauer theory for algebras graded by abelian groups.

• In his positive solution to the Specht problem for associative algebras over a field of characteristic zero, Kemer [1990] needed to describe all gradings on $M_2(k)$ by the cyclic group C_2.

• Gradings on matrix algebras and on certain structural matrix algebras are used in the study of numerical invariants of PI algebras.

• C_2-gradings on a matrix algebra are the superalgebra structures on matrices.

In *D, Ion, Năstăsescu, Rios* [1999] gradings on $M_n(k)$ for which any matrix unit e_{ij} is a homogeneous element were studied; such gradings were called good gradings.

In some cases, any G-grading on $A = M_n(k)$ is isomorphic to a good grading, for example if one of the conditions holds:

- There exists a graded A-module which is simple as an A-module.
- G is torsionfree.
- One of the matrix units e_{ij} is a homogeneous element.

Let $V = \oplus_{g \in G} V_g$ be a G-graded vector space of dimension n. Then the algebra $\mathrm{End}(V)$ has a G-grading given by

$$\mathrm{End}(V)_\sigma = \{f \in \mathrm{End}(V) \mid f(V_g) \subset V_{\sigma g} \text{ for any } g \in G\}.$$

Denote by $\mathrm{END}(V)$ the G-graded algebra obtained in this way.

It was explained that any good G-grading on $M_n(k)$ is isomorphic to a graded algebra of the form $\mathrm{END}(V)$, where V is n-dimensional and G-graded; also, any graded algebra of the type $\mathrm{END}(V)$ is isomorphic to $M_n(k)$ with a certain good grading.

Thus instead of classifying good G-gradings on $M_n(k)$, we can classify graded algebras of the type $\mathrm{END}(V)$, where V is a G-graded vector space of dimension n.

If V is a G-graded vector space, and $\sigma \in G$, let $V(\sigma)$ be the G-graded vector space such that $V(\sigma) = V$ as a vector space, with the grading shifted by σ, i.e. $V(\sigma)_g = V_{g\sigma}$ for any $g \in G$.

It was proved in *Caenepeel, D, Năstăsescu* [2002]

Theorem. *If V and W are G-graded vector spaces of dimension n, then $\mathrm{END}(V) \simeq \mathrm{END}(W)$ if and only if $W \simeq V(\sigma)$ for some $\sigma \in G$.*

Corollary. *Good G-gradings on $M_n(k)$ are classified by the orbits of the right biaction of S_n (by permutations) and G (by right translations) on G^n.*

Theorem. *If k is algebraically closed, then any C_m-grading on $M_n(k)$ is isomorphic to a good grading.*

Descent theory and some related results of *Caenepeel, D, Le Bruyn* [1999] were used to prove:

Theorem. *Let k be a field and let G be an abelian group. If V is a G-graded $\overline{k}$-vector space, then the forms of the good G-grading $\mathrm{END}(V)$ on $M_n(\overline{k})$ (i.e the G-gradings on $M_n(k)$ such that $\overline{k} \otimes_k M_n(k) \simeq \mathrm{END}(V)$ as G-graded $\overline{k}$-algebras) are in bijection to the Galois extensions of k with Galois group $\mathcal{I}(V) = \{\sigma \in G | V(\sigma) \simeq V\}$.*

Bahturin, Seghal and Zaicev [2001], described all gradings on $M_n(k)$ by abelian groups G, in the case where k is algebraically closed of characteristic 0. The result was extended to gradings by arbitrary groups, for any algebraically closed k, in *Bahturin, Zaicev* [2002], [2003].

A grading is called a fine grading if the dimension of any homogeneous component is at most 1. A special type of fine grading is obtained as follows. Let n be a positive integer and ε a primitive nth root of unity in k. Consider the matrices in $M_n(k)$

$$X = \begin{pmatrix} \varepsilon^{n-1} & 0 & \dots & 0 \\ 0 & \varepsilon^{n-2} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 1 \end{pmatrix}, Y = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 1 & 0 & 0 & \dots & 0 \end{pmatrix}$$

Then

$$XY = \varepsilon YX, X^n = I_n, Y^n = I_n$$

and $\{X^i Y^j \mid 0 \leq i, j \leq n-1\}$ is linearly independent, so $A = M_n(k)$ has a $C_n \times C_n =< g > \times < h >$-grading given by $A_{g^i h^j} = kX^i Y^j$ for any $0 \leq i, j \leq n-1$. Denote this graded algebra by $A(n, \varepsilon)$.

Assume that k is algebraically closed of characteristic 0, and consider gradings by abelian groups G. The results of BSZ are:

Theorem I *Any G-grading on $A = M_n(k)$ is isomorphic to one of the form $B \otimes C$, where B is a matrix algebra with a good grading, and C is a matrix algebra with a fine grading.*

Theorem II *Any fine grading on a matrix algebra is isomorphic to*

$$A(n_1, \varepsilon_1) \otimes \ldots \otimes A(n_r, \varepsilon_r)$$

for some $r, n_1, \ldots, n_r, \varepsilon_1, \ldots, \varepsilon_r$.

A proof of Theorem I.

Based on ideas appearing in *D, Ion, Năstăsescu, Rios* [1999], *Caenepeel, D, Năstăsescu* [2002], and a graded version of the density theorem proved in *Gomez Pardo, Năstăsescu* [1991]; the result is contained in a structure result for graded simple algebras in the book of *Năstăsescu, Van Oystaeyen* [2004], without mentioning the interest for gradings on matrix algebras.
A similar proof is given in *Elduque, Kochetov* [2013], where the gradings are described and classified.

Let k be a field (not necessarily algebraically closed), and let G be a group (not necessarily abelian). If $A = M_n(k)$ has a G-grading, let Σ be a gr-simple A-module, i.e. a simple object in the category of G-graded left A-modules. Let $\Delta = \mathrm{End}_A(\Sigma)$, which has a G-grading given by

$$\Delta_g = \{f \in \mathrm{End}_A(\Sigma) |\ f(\Sigma_h) \subseteq \Sigma_{hg} \text{ for any } h \in G\}$$

Then Δ is a G-graded division algebra (i.e. any non-zero homogeneous element is invertible), and if S is a simple A-module, then $\Sigma \simeq S^m$ for some positive integer m, so $\Delta \simeq \mathrm{End}_A(S^m) \simeq M_m(k)$.

Moreover, Σ is a left A, right Δ graded bimodule.

In a similar manner, $\mathrm{End}(\Sigma_\Delta)$ is also equipped with a G-graded algebra structure, and one has a morphism of graded algebras

$$\phi : A \to \mathrm{End}(\Sigma_\Delta), \phi(a)(x) = ax.$$

By a graded version of the density theorem, ϕ is surjective, thus also bijective (since A is a simple algebra). We obtain that

$$A \simeq \mathrm{End}(\Sigma_\Delta)$$

Since Δ is a graded division algebra, Σ is a free Δ-module with a homogeneous basis, thus $\Sigma \simeq V \otimes \Delta$ for some G-graded vector space V.

If G is abelian, then

$$\mathrm{End}(\Sigma_\Delta) \simeq \mathrm{END}(V) \otimes \Delta$$

as G-graded algebras. Thus any grading on $M_n(k)$ by an abelian group is the tensor product of a good grading and a graded division algebra (on certain matrix algebras).
If k is algebraically closed, Δ_e is a finite extension of k, so $\Delta_e = k$; then all the homogeneous components of Δ have dimensions at most 1, so Δ has a fine grading; this is just Theorem I.

If G is not necessarily abelian, let $\sigma_1, \ldots, \sigma_r$ the degrees of the elements in a homogeneous Δ-basis of Σ. Then we get that $A \simeq M_r(\Delta)$ as graded algebras, where the grading on $M_r(\Delta)$ is given by

$$M_r(\Delta)(\sigma_1, \ldots, \sigma_r)_g = \begin{pmatrix} \Delta_{\sigma_1 g \sigma_1^{-1}} & \Delta_{\sigma_1 g \sigma_2^{-1}} & \ldots & \Delta_{\sigma_1 g \sigma_r^{-1}} \\ \Delta_{\sigma_2 g \sigma_1^{-1}} & \Delta_{\sigma_2 g \sigma_2^{-1}} & \ldots & \Delta_{\sigma_2 g \sigma_r^{-1}} \\ \ldots & \ldots & \ldots & \ldots \\ \Delta_{\sigma_r g \sigma_1^{-1}} & \Delta_{\sigma_r g \sigma_2^{-1}} & \ldots & \Delta_{\sigma_r g \sigma_r^{-1}} \end{pmatrix}$$

In conclusion, describing gradings (by arbitrary groups) on matrix algebras (over arbitrary fields) reduces to finding all graded division algebra structures on matrix algebras.

Gradings on diagonal algebras

Let $A = k^n$. If k is algebraically closed, *Bichon* [2008] described gradings on A, by considering coactions of Hopf algebras on A and using an approach of Manin and Wang to show that there exists a Hopf algebra coaction on the diagonal algebra k^n, which is universal in a large class of Hopf algebras.

A different approach was used in *D* [2007] for describing all gradings on A for any field.

A grading on A is called:

- *faithful* if $\mathrm{supp}(A)$ generates the group G.
- *ergodic* if $\dim(A_e) = 1$.

Ergodic gradings are classified by the following.

Theorem. *Let $A = k^n$. Then the following assertions hold.*
(1) If $\mathrm{char}(k)|n$, then there do not exist ergodic group gradings on A.
(2) If $\mathrm{char}(k)$ does not divide n, then the faithful ergodic group gradings on A are by abelian groups H of order n, such that k contains a primitive e-th root of unity, where e is the exponent of H. For such an H, any faithful ergodic H-grading on A is isomorphic to the group algebra kH with the usual H-grading.

The following shows that a faithful group grading on a diagonal algebra is some sort of a direct sum of ergodic gradings.
If M is a non-empty subset of $\{1, \ldots, n\}$, we denote by $A_M = \sum_{j \in M} ke_j$; clearly $A_M \simeq k^{|M|}$.

Theorem. *Let k be a field and let n be a positive integer. If $A = \oplus_{g \in G} A_g$ is a faithful grading on $A = k^n$ by the group G, then there exist*

- *Abelian groups $H_1, \ldots, H_s$ of exponents $e_1, \ldots, e_s$, such that $|H_1| + \ldots + |H_s| = n$, and k contains a primitive e_i-th root of unity for any $1 \leq i \leq s$;*
- *A surjective group morphism $\phi : H_1 * \ldots * H_s \to G$ such that $\phi(H_i) \simeq H_i$ for any $1 \leq i \leq s$ (for simplicity we identify $\phi(H_i)$ and H_i);*
- *A partition $M_1, \ldots, M_s$ of the set $\{1, \ldots, n\}$ such that $|M_i| = |H_i|$;*
- *An ergodic H_i-grading on the algebra A_{M_i} for any $1 \leq i \leq s$,*

such that $\mathrm{supp}(A) = H_1 \cup \ldots \cup H_s$ and $A_g = \sum_{1 \leq i \leq s} (A_{M_i})_g$ for any $g \in \mathrm{supp}(A)$ (where we regard the H_i-grading of A_{M_i} as a G-grading).

Conversely, for any abelian groups $H_1, \ldots, H_s$*, any group morphism* $\phi : H_1 * \ldots * H_s \to G$*, and any partition* $M_1, \ldots, M_s$*, satisfying conditions as above, a faithful* G*-grading on* k^n *can be constructed by putting together ergodic* H_i*-gradings of* $A_{M_i} \simeq k^{|H_i|}$ *for all* $1 \leq i \leq s$ *as a direct sum as above.*

Gradings on upper block triangular matrix algebras

Let $A = M(\rho, k)$ be the algebra

$$\begin{pmatrix} M_{m_1}(k) & M_{m_1,m_2}(k) & \dots & M_{m_1,m_r}(k) \\ 0 & M_{m_2}(k) & \dots & M_{m_2,m_r}(k) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & M_{m_r}(k) \end{pmatrix}$$

of upper block triangular matrices. Gradings on A are classified by

• *Valenti, Zaicev*[2012] for gradings by abelian groups, for algebraically closed k of characteristic 0.

• *Kotchetov, Yasumura* for gradings by abelian groups and arbitrary algebraically closed k.

• *Yasumura* [2018] for gradings by arbitrary groups and algebraically closed k of characteristic 0 or characteristic $> \dim A$.

Good gradings on structural matrix algebras

Joint work with Filoteia Beșleagă.

Let A be a structural matrix algebra over k. It is associated with a preorder relation ρ on the set $\{1, \dots, n\}$; A consists of all matrices $(a_{ij})_{1 \le i,j \le n}$ such that $a_{ij} = 0$ whenever $(i,j) \notin \rho$. We denote $A = M(\rho, k)$; in other terminology, this is the incidence algebra over k associated with ρ.

PROBLEM. *Classify all gradings on $A = M(\rho, k)$ such that each e_{ij} with $i\rho j$ is a homogeneous element (these are called good gradings).*

Let $\sim$ be the equivalence relation on $\{1, \dots, n\}$ associated with ρ, i.e. $i \sim j$ if and only if $i \rho j$ and $j \rho i$, and let $\mathcal{C}$ be the set of equivalence classes. Then ρ induces a partial order $\leq$ on $\mathcal{C}$ defined by $\hat{i} \leq \hat{j}$ if and only if $i \rho j$, where $\hat{i}$ denotes the equivalence class of i.
For any $\alpha \in \mathcal{C}$, let m_α be the number of elements of α.

Definition. *A ρ-flag is an n-dimensional vector space V with a family $(V_\alpha)_{\alpha \in \mathcal{C}}$ of subspaces such that there is a basis B of V and a partition $B = \bigcup_{\alpha \in \mathcal{C}} B_\alpha$ with the property that $|B_\alpha| = m_\alpha$ and $\bigcup_{\beta \leq \alpha} B_\beta$ is a basis of V_α for any $\alpha \in \mathcal{C}$.*
If $\mathcal{F} = (V, (V_\alpha)_{\alpha \in \mathcal{C}})$ and $\mathcal{F}' = (V', (V'_\alpha)_{\alpha \in \mathcal{C}})$ are ρ-flags, then a morphism of ρ-flags from $\mathcal{F}$ to $\mathcal{F}'$ is a linear map $f : V \to V'$ such that $f(V_\alpha) \subset V'_\alpha$ for any $\alpha \in \mathcal{C}$.

Example

If $A = M(\rho, k)$ is the algebra

$$\begin{pmatrix} M_{m_1}(k) & M_{m_1,m_2}(k) & \dots & M_{m_1,m_r}(k) \\ 0 & M_{m_2}(k) & \dots & M_{m_2,m_r}(k) \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & M_{m_r}(k) \end{pmatrix}$$

of upper block triangular matrices, with diagonal blocks of size $m_1, \dots, m_r$, then ρ is such that $\mathcal{C} = \{\alpha_1, \dots, \alpha_r\}$ is totally ordered, say $\alpha_1 < \dots < \alpha_r$, and $|\alpha_i| = m_i$ for any $1 \leq i \leq r$. A ρ-flag is a usual flag of signature $(m_1, \dots, m_r)$.

Proposition. *Let* $\mathcal{F} = (V, (V_\alpha)_{\alpha \in \mathcal{C}})$ *be a* ρ*-flag. Then the algebra* $\mathrm{End}(\mathcal{F})$ *of endomorphisms of* $\mathcal{F}$ *is isomorphic to* $M(\rho, k)$.

An application of this description is the computation of the automorphism group of a structural matrix algebra. The steps are:

• The $\mathrm{End}(\mathcal{F})$-submodules of V are in a bijective correspondence with the antichains of $\mathcal{C}$; let $\mathcal{A}(C)$ be the lattice structure on the set of all such antichains, induced via this bijection.

• An algebra automorphism $\varphi : \mathrm{End}(\mathcal{F}) \to \mathrm{End}(\mathcal{F})$ induces a linear isomorphism $\gamma : V \to V$ which is a φ'-isomorphism for a certain deformation φ' (also an algebra automorphism) of φ.

• γ induces an automorphism of the lattice of $\mathrm{End}(\mathcal{F})$-submodules of V, thus also an automorphism of the lattice $\mathcal{A}(C)$. Such an automorphism is completely determined by an automorphism g of the poset $\mathcal{C}$.

• φ can be recovered from g, the deformation constants producing φ' from φ, and a matrix of γ in a fixed pair of bases.

Define

$$\mathrm{Aut}_0(\mathcal{C},\leq) = \{g \in \mathrm{Aut}(\mathcal{C},\leq) \mid m_\alpha = m_{g(\alpha)} \text{ for any } \alpha \in \mathcal{C}\}$$

$$\mathcal{T} = \{(a_{ij})_{i\rho j} \subset k^* \mid a_{ij}a_{jr} = a_{ir} \text{ for any } i,j,r \text{ with } i\rho j, j\rho r\}$$

The automorphism group of a structural matrix algebra is described by

Theorem.

$$\mathrm{Aut}(\mathrm{End}(\mathcal{F})) \simeq \frac{U(M(\rho,k)) \rtimes (\mathrm{Aut}_0(\mathcal{C}) \ltimes \mathcal{T})}{D},$$

where

$D = \{diag(d_1,\ldots,d_n) \rtimes (Id \ltimes (d_i^{-1}d_j)_{i\rho j}) \mid d_1,\ldots,d_n \in k^*\}.$

Another description, previously given by *Coelho* [1993], can be derived.

Back to good gradings on $M(\rho, k)$

A G-graded ρ-flag is a ρ-flag $(V, (V_\alpha)_{\alpha\in\mathcal{C}})$ such that V is a G-graded vector space, and the basis B from the definition of a ρ-flag consists of homogeneous elements.

If $\mathcal{F} = (V, (V_\alpha)_{\alpha\in\mathcal{C}})$ is a G-graded ρ-flag, then $\mathrm{End}(\mathcal{F})$ is a G-graded algebra, with the grading given by

$$\mathrm{End}(\mathcal{F})_\sigma = \{f \in \mathrm{End}(\mathcal{F}) \mid f(V_g) \subseteq V_{\sigma g} \text{ for any } g \in G\}.$$

Denote it by $\mathrm{END}(\mathcal{F})$; it is isomorphic to a good grading on $M(\rho, k)$.

Question. *Do all good gradings on $M(\rho, k)$ arise in this way?*

Giving a good G-grading on $M(\rho, k)$ is equivalent to giving a family $(u_{ij})_{i\rho j}$ of elements of G such that $u_{ij}u_{jr} = u_{ir}$ for any i, j, r with $i\rho j$ and $j\rho r$. Regard such a family as a function $u : \rho \to G$, defined by $u(i, j) = u_{ij}$ for any i, j with $i\rho j$; we call u a transitive function on ρ with values in G.

Examples of a transitive functions on ρ can be obtained as follows. Let $g_1, \ldots, g_n \in G$, and let $u_{ij} = g_i g_j^{-1}$ for any i, j with $i\rho j$. Then $(u_{ij})_{i\rho j}$ is a transitive function on ρ. A transitive function on ρ is called trivial if it is obtained in this way.

We associate with ρ the graph $\Gamma = (\Gamma_0, \Gamma_1)$ whose set Γ_0 of vertices is the set $\mathcal{C}$ of equivalence classes. The set Γ_1 of arrows is constructed as follows: if $\alpha, \beta \in \mathcal{C}$, there is an arrow from α to β if $\alpha < \beta$ and there is no $\gamma \in \mathcal{C}$ with $\alpha < \gamma < \beta$.

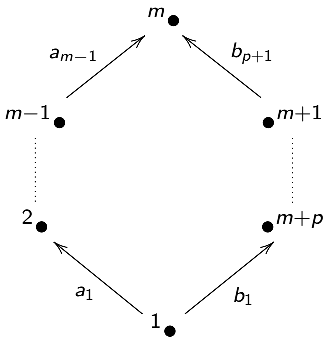
Proposition. *Let G be a group. The following are equivalent:*
(1) *Any good G-grading on $M(\rho, k)$ arises from a graded flag.*
(2) *Any transitive function $u : \rho \to G$ is trivial.*
(3) *Any transitive function $w : \leq \to G$ is trivial, where $\leq$ is the partial order on $\mathcal{C}$.*
(4) *For any function $v : \Gamma_1 \to G$ such that $v(a_1) \dots v(a_r) = v(b_1) \dots v(b_s)$ for any paths $a_1 \dots a_r$ and $b_1 \dots b_s$ in Γ starting from the same vertex and terminating at the same vertex, there exists a function $f : \Gamma_0 \to G$ such that $v(a) = f(s(a))f(t(a))^{-1}$ for any $a \in \Gamma_1$.*

Let $F(\Gamma)$ be the free group generated by the set Γ_1 of arrows of Γ. Let $A(\Gamma)$ be the subgroup of $F(\Gamma)$ generated by all elements of the form $a_1 \dots a_r b_p^{-1} \dots b_1^{-1}$, where $a_1 \dots a_r$ and $b_1 \dots b_p$ are two paths (in Γ) starting from the same vertex and terminating at the same vertex.
We also consider the subgroup $B(\Gamma)$ of $F(\Gamma)$ generated by all elements of the form $a_1 a_2^{\varepsilon_2} \dots a_m^{\varepsilon_m}$, where $a_1, \dots, a_m$ are arrows forming in this order a cycle in the undirected graph obtained from Γ when omitting the direction of arrows, and $\varepsilon_i = 1$ if a_i is in the direction of the directed cycle given by a_1, and $\varepsilon_i = -1$ otherwise. Clearly $A(\Gamma) \subseteq B(\Gamma)$.

Proposition. *The following are equivalent.*
(1) *For any group G, any transitive function $u : \rho \to G$ is trivial.*
(2) $A(\Gamma)^N = B(\Gamma)^N$.
(3) *Any generator b of $B(\Gamma)$ can be written in the form $b = g_1 x_1 g_1^{-1} \dots g_m x_m g_m^{-1}$ for some positive integer m, some $g_1, \dots, g_m \in F(\Gamma)$ and some $x_1, \dots, x_m$ among the generators in the construction of $A(\Gamma)$.*

Example.

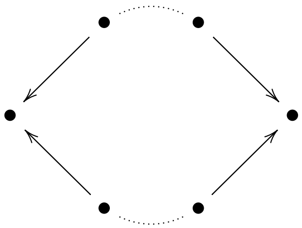
Assume that ρ is a preorder relation such that the associated graph Γ is of the form



for some integers $m \geq 3$ and $p \geq 1$. Then for any group G, any transitive function $u : \rho \to G$ is trivial.

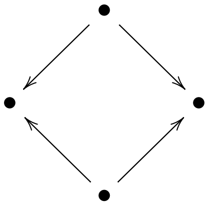
Example.

Assume that ρ is a preorder relation such that the associated graph Γ is of the form



Thus the un-directed graph Γ^u associated to Γ is cyclic, and in Γ there are at least two vertices where both adjacent arrows terminate (equivalently, Γ^u is cyclic and Γ is not of the type in the previous example. Then for any non-trivial group G, there exist transitive functions $u : \rho \to G$ that are not trivial.

The simplest example of such a graph is

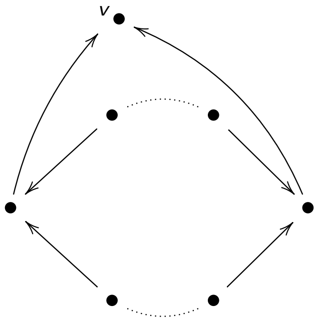


and the corresponding structural matrix algebra, whose not all good gradings arise from graded flags, is

$$\begin{pmatrix} k & 0 & k & k \\ 0 & k & k & k \\ 0 & 0 & k & 0 \\ 0 & 0 & 0 & k \end{pmatrix}.$$

Example.

If the corresponding graph is



then all transitive functions (on the corresponding preordered set) are trivial.

Classification of gradings of the type $\mathrm{END}(\mathcal{F})$

Let $\mathcal{C} = \mathcal{C}^1 \cup \ldots \cup \mathcal{C}^q$ be the decomposition of $\mathcal{C}$ in disjoint connected components; these correspond to the connected components of the undirected graph Γ^u. For each $1 \leq t \leq q$, let ρ_t be the preorder relation on the set $\bigcup_{\alpha \in \mathcal{C}^t} \alpha$, by restricting ρ.

If $V^t = \sum_{\alpha \in \mathcal{C}^t} V_\alpha$, then $\mathcal{F}^t = (V^t, (V_\alpha)_{\alpha \in \mathcal{C}^t})$ is a G-graded ρ_t-flag with basis $\bigcup_{\alpha \in \mathcal{C}^t} B_\alpha$. Obviously, $V = \bigoplus_{1 \leq t \leq q} V^t$. In a formal way we can write $\mathcal{F} = \mathcal{F}^1 \oplus \ldots \oplus \mathcal{F}^q$, where $\mathcal{F}$ is a G-graded ρ-flag, and $\mathcal{F}^t$ is a G-graded ρ_t-flag for each $1 \leq t \leq q$.

Definition. *Let ρ and μ be isomorphic preorder relations (i.e. the preordered sets on which ρ and μ are defined are isomorphic). Let $\mathcal{C}$ and $\mathcal{D}$ be the posets associated with ρ and μ, and let $g : \mathcal{C} \to \mathcal{D}$ be an isomorphism of posets. We say that a ρ-flag $\mathcal{F} = (V, (V_\alpha)_{\alpha \in \mathcal{C}})$ is g-isomorphic to a μ-flag $\mathcal{G} = (W, (W_\beta)_{\beta \in \mathcal{D}})$ if there is a linear isomorphism $u : V \to W$ such that $u(V_\alpha) = W_{g(\alpha)}$ for any $\alpha \in \mathcal{C}$.*
If $\mathcal{F}$ and $\mathcal{G}$ are G-graded flags, we say that they are g-isomorphic as graded flags if there is such an u which is a morphism of graded vector spaces.

Theorem. *Let* $\mathcal{F} = (V, (V_\alpha)_{\alpha\in\mathcal{C}})$ *and* $\mathcal{F}' = (V', (V'_\alpha)_{\alpha\in\mathcal{C}})$ *be* G*-graded* ρ*-flags. Then the following assertions are equivalent:*
(1) $\mathrm{END}(\mathcal{F})$ *and* $\mathrm{END}(\mathcal{F}')$ *are isomorphic as* G*-graded algebras.*
(2) *There exist* $g \in \mathrm{Aut}_0(\mathcal{C})$, $\sigma_1, \ldots, \sigma_q \in G$ *and a* g*-isomorphism* $\gamma : V \to V'$ *between the (ungraded)* ρ*-flags* $\mathcal{F}$ *and* $\mathcal{F}'$*, such that* $\gamma_{|V^t}^{|V'^{\overline{g}(t)}} : V^t \to V'^{\overline{g}(t)}$ *is a linear isomorphism of left degree* σ_t *for any* $1 \leq t \leq q$*, where* $\overline{g} \in S_q$ *is the permutation induced by* g*, i.e.* $g(\mathcal{C}^t) = \mathcal{C}^{\overline{g}(t)}$.
(3) *There exists a permutation* $\tau \in S_q$*, an isomorphism* $g_t : \mathcal{C}^t \to \mathcal{C}^{\tau(t)}$ *for each* $1 \leq t \leq q$*, and* $\sigma_1, \ldots, \sigma_q \in G$*, such that* $\mathcal{F}^t(\sigma_t)$ *is* g_t*-isomorphic to* $\mathcal{F}'^{\tau(t)}$ *for any* $1 \leq t \leq q$.

Theorem. *The isomorphism types of G-gradings of the type $\mathrm{END}(\mathcal{F})$, where $\mathcal{F}$ is a G-graded ρ-flag, are classified by the orbits of the right action of the group $\prod_{\alpha\in\mathcal{C}} S(\alpha) \rtimes (\mathrm{Aut}_0(\mathcal{C}) \ltimes G^q)$ on the set G^n.*

If ρ is a partial order, then all good gradings are classified in *Beşleagă, D, van Wyk* [2018].

Theorem. *Let G be a group. If $(u_{ij})_{i\rho j}$ and $(v_{ij})_{i\rho j}$ are two G-valued transitive functions on ρ, then the corresponding good G-gradings on $A = M(\rho, k)$ are isomorphic if and only if there exists an automorphism φ of the poset $(\{1, \dots, n\}, \rho)$ such that $v_{ij} = u_{\varphi(i)\varphi(j)}$ for any i, j with $i\rho j$.*
Thus the isomorphism types of good G-gradings on $A = M(\rho, k)$ are in bijection to the orbits of the right action of $\mathrm{Aut}(\rho)$ on $T(\rho, G)$.